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BOUNDARY VALUE PROBLEM WITH SHIFT FOR TWO SIMPLE CONNECTED REGIONS

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Abstract

In this paper it is considered a boundary value problem for two analytic functions in different simple connected regions. If index corresponded to the problem is not greater then zero that it is shown solvable of that problem without any conditions and its solution is given by (4) in explicite form. If however the index is greater then zero then problem is solvable and its solution is given by (5) if real conditions of solvability (6) hold.

1. Let S and D are finite simple–connected regions, bounded by closed Ljapunov curves L and Γ respectively. Suppose that the contours L and Γ are traversed in the positive sense to their interiors S and D respectively, so that a person moving around L or Γ in this direction always has their interiors lying to his left.

Let $\alpha(t)$ be the function given on L satisfying the following conditions:

a) It transforms homeomorphically the closed contour L into the closed contour changing the direction of movement.

b) Function $\alpha(t)$ has continuous derivatives which are different from zero at all the points of the contour L.

Let the function $\alpha^{-1}(t)$, $t \in L$, be the inverse function of $\alpha(t)$.

We shall determine the functions $\phi_1(z)$ and $\phi_2(z)$ which are analytic in S and D respectively, whose boundary values on the appropriate contours satisfy the following boundary conditions:

$$\phi_2\big(\alpha(t)\big) = G(t) * \phi_1(t) + g(t)\,, \quad t \in L\,, \tag{1}$$

where $G(t)$ and $g(t)$ are continues function on L in the sense of Holder ([5]).

In the case D = exterior (s) many authors (cf. [4]) considered the problem of finding functions $\phi_1(z)$ and $\phi_2(z)$ which are analytic in S and D, respectively whose boundary values on appropriate contours satisfy the boundary condition analogous (1).

Assume that the coordinate origin belongs to the regions S and define the function $G_{(2)}(t)$ in the following way: $G_{(2)}(t) = f^{-k}G(t)$, $t \in L$, where is $k = \frac{[\arg G(t)]_L}{2}$ the index corresponded to the function $G(t)$.

Now, $\frac{\arg G_{(2)}(t)]_L}{2} = 0$. From the paper [2] it follows that for the homogeneous boundary value problem with the coefficient $G_{(2)}(t)$ there exist the functions $X_{(2)}(t)$ and $X_{(2),1}(t)$ being analytic in S and D respectively, and different from zero successively in $S \cup L$ and $D \cup \Gamma$, and which on the appropriate contours L and Γ have the limits $X_{(2)}(t) \in H(L)$ and $X_{(2),1}(t) \in H(L)$ satisfying the following boundary value condition: $X_{(2),1}\big(\alpha(t)\big) = G_{(2)}(t) * X_{(2)}(t)$, $t \in L$.

Those functions are determined by the formulae:

$$X_{(2)}(z) = \exp\left(-\frac{1}{2\pi i}\int_L \frac{h(t)}{t-z}\,dt\right), \qquad z \in S,$$

$$X_{(2),1}(z) = \exp\left(\frac{1}{2\pi i}\int_\Gamma \frac{h\big(\alpha^{-1}(t)\big)}{t-z}\,dt\right), \qquad z \in D,$$

where $h(t)$, $t \in L$, is the solution of the equation:

$$(Fh)(t) \equiv h(t) - \frac{1}{2\pi i}\int_L \left(\frac{1}{t-\sigma} - \frac{h'(\sigma)}{h(\sigma)-h(t)}\right) h(\sigma)d\sigma = \\ = \ln\, G_{(2)}(t), \qquad t, \sigma \in L.$$

According to all of this it follows that the coefficient $G(t)$ of the problem (1) for the contour L, can be represented in the form:

$$G(t) = \frac{X_{0,1}\big(\alpha(t)\big)}{(t^{-k}X_0(t)}, \quad t \in L. \tag{2}$$

In this way, the boundary condition (1) can be written in the following way:

$$\frac{\phi_2\big(\alpha(t)\big)}{X_{0,1}\big(\alpha(t)\big)} - \frac{\phi_1(t)}{t^{-k}X_0(t)} = \frac{g(t)}{X_{0,1}\big(\alpha(t)\big)}, \quad t \in L. \tag{3}$$

Let us denote by $f_2(z)$ the function $\frac{\phi_2(z)}{X_{0,1}(z)}$, $z \in D$. In the case $k < 0$ function $\frac{\phi_1(z)}{z^{-k}X_0(z)}$, $z \in S$, has the point $z = 0$ as a pole of order $-k$, so that it can be represented in the form:

$$\frac{\phi_1(z)}{z^{-k}X_0(z)} = \sum_{i=1}^{-k} \frac{c_i}{z^i} + f_1(z), \quad z \in S,$$

where $f_1(z)$ is indefinite analytic function in S and c_i, $i = 1, 2, \ldots, -k$, are complex constants.

If we introduce notation $B_{2j-1} = Re\, c_j$, $B_{2j} = Im\, c_j$, $g_{2j-1}(t) = \frac{1}{t^j}$, $t \in L$, $g_{2j}(t) = \frac{i}{t^j}$, $t \in L$ we shall get the boundary value problem:

$$f_2\big(\alpha(t)\big) - f_1(t) = \sum_{j=1}^{-2k} B_j g_j(t)$$

with solution:

$$f_1(z) = B_{(2)} - \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_L \frac{\varphi_j(t)}{t-z}\, dt, \qquad z \in S,$$

$$f_2(z) = B_{(2)} + \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_\Gamma \frac{\varphi_j\big(\alpha^{-1}(t)\big)}{t-z}\, dt, \quad z \in D,$$

where $B_{(2)}$ is an arbitrary complex constant and $\varphi_j(t)$ are the solutions of Fredholm's integral equations: $(F\varphi_j)(t) = g_j(t)$, $j = 1, 2, \ldots, -2k$.

Let us assume that $B_{-2k+1} = Re\, B_0$, $B_{-2k+2} = Im\, B_0$ and let us define the functions:

$$W_{2j-1}(z) = \frac{1}{z^j} - \frac{1}{2\pi i} \int_L \frac{\varphi_{2j-1}(t)}{t-z}\, dt, \quad z \in S,$$

$$W_{2j}(z) = \frac{1}{z^j} - \frac{1}{2\pi i} \int_L \frac{\varphi_{2j}(t)}{t-z}\, dt, \quad z \in S,$$

$$j = 1, 2, \ldots, -k,$$

$$W_{-2k+1}(z) = 1, \quad z \in S, \qquad W_{-2k+2}(z) = i, \quad z \in S$$

$$V_j(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi_j\big(\alpha^{-1}(t)\big)}{t-z}\, dt, \quad z \in D,$$

$$V_{-2k+1}(z) = 1, \quad z \in D \qquad V_{-2k+2}(z) = i, \quad z \in D.$$

Now, we can give the general solution of the boundary value problem (1) in the case $k < 0$ in the following way:

$$\begin{aligned}\phi_1(z) &= z^{-k}X_{(2)}(z)\left(\sum_{j=1}^{-2k+2} B_jW_j - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt\right), \quad z \in S,\\ \phi_2(z) &= X_{(2),1}(z)\left(\sum_{j=1}^{-2k+2} B_jV_j(z) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)}{t-z}\,dt\right), \quad z \in D,\end{aligned} \tag{4}$$

where $\varphi(t)$ is the solutions of the integral equation:

$$F\varphi(t) = \frac{g(t)}{X_{0,1}\left(\alpha(t)\right)}.$$

In the case $k > 0$, the solution, if it exists at all, may be represented in the form:

$$\begin{aligned}\phi_1(z) &= z^{-k}X_{(2)}(z)\left(C_{(2)} - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt\right) \quad z \in S,\\ \phi_2(z) &= X_{(2),1}(z)\left(C_{(2)} + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi\left(\alpha^{-1}(t)\right)}{t-z}\,dt\right) \quad z \in D,\end{aligned} \tag{5}$$

where $C_{(2)}$ is an arbitrary complex constant.

For the purpose of finding the conditions of solvability, we shall expand the function:

$$\phi_*(z) = C_{(2)} - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt$$

in Taylor's series in the neighborhood of the point $z = 0$. Thus, the function:

$$\phi_1(z) = z^{-k}X_{(2)}(z)\left(C_{(2)} - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt - \sum_{j=1}^{\infty}\frac{z^j}{2\pi i}\int_L \frac{\varphi(t)}{t^{j+1}}\,dt\right), \quad z \in S,$$

will be analytic in S if all the coeficients at $z_{(2)}, z_1, \ldots, z^{k-1}$ in this expansion are equal to zero. If we choose:

$$C_{(2)}\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt$$

and separate the real and imaginary parts in the conditions:

$$\int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0\,, \quad j = 1, 2, \ldots, k-1\,,$$

we get the next $2(k-1)$ real conditions of solvability:

$$Re \int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0, \qquad Im \int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0\,, \quad j = 1, 2, \ldots, k-1\,. \tag{6}$$

The unique solution in this case is given by (5). With this we have proved the following theorem.

Theorem 1. *It the index* $k = \frac{[\arg G(t)]_L}{2} > 0$ *then the problem* (1) *is solvable and its unique solution is given by* (5) *if and only if all the* $2(k-1)$ *real conditions of solvability* (6) *hold. If, however, index* $k <= 0$ *then the boundary value problem* (1) *is solvable and its solution, represented by the formula* (4), *contains* $2(-k+1)$ *arbitrary real constants.*

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ГРАНИЧЕН ПРОБЛЕМ СО ПОМЕСТУВАЊЕ ЗА ДВЕ ПРОСТО ПОВРЗАНИ ОБЛАСТИ

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Резиме

Во оваа работа е разгледуван граничен проблем за две аналитички функции во две различни просто поврзани области. Ако индексот на проблемот не е поголем од нула, проблемот е решен без дополнителни услови и неговото решение е дадено со (4). Ако индексот е поголем од нула, проблемот се решава при претпоставка да важат условите (6) и притоа решението е дадено со (5).

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