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ON CAUCHY-BUNIAKOWSKI-SCHWARZ'S INEQUALITY FOR REAL NUMBERS

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Abstract. A new refinement of the well known Cauchy-Buniakowski-Schwarz's inequality for real numbers is given.

The following inequality is known in literature as Cauchy-Buniakowski-Schwarz's inequality:

$$\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 \geq \left(\sum_{i\in I} p_i a_i b_i\right)^2 \tag{1}$$

where $(a_i)_{i\in N}$, $(b_i)_{i\in N}$ are sequences of real numbers, $(p_i)_{i\in N}$ are positive real numbers and I is a finite part of N. Note that the equality holds in (1) iff $a_i = r\cdot b_i$ for all $i\in I$, where r is an arbitrary real number.

In paper [8] was proved the following refinement of (1):

$$\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2 \geq$$

$$\geq \left|\sum_{i\in I} p_i a_i |a_i| \sum_{i\in I} p_i b_i |b_i| - \sum_{i\in I} p_i |a_i| b_i \sum_{i\in I} p_i a_i |b_i|\right| \geq 0$$

A property of monotonicity for the inequality (1) is embodied in the following (see [13]):

$$\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right]^{1/2} - \left|\sum_{i\in I} p_i a_i b_i\right| \geq$$

$$\geq \left[\sum_{i\in I} q_i a_i^2 \sum_{i\in I} q_i b_i^2\right]^{1/2} - \left|\sum_{i\in I} q_i a_i b_i\right| \geq 0,$$

where $p_i \geq q_i \geq 0$ $(i\in N)$ and $(a_i)_{i\in N}$, $(b_i)_{i\in N}$ and I are as above. Note that the above inequality was proved in [13] for $I=\{1,\ldots,n\}$, but a similar argument for I a finite part of N also holds. We will omit the details.

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The main aim of this paper is to give other improvements for (1).

Theorem. Let $(a_i)_{i\in N}$, $(b_i)_{i\in N}$ and $(p_i)_{i\in N}$ be sequences of real numbers so that $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ $(i,j\in N)$ and $p_i > 0$ for all $i\in N$. Then for all H a finite part of N are has the inequality:

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i a_i b_i) \geq \max\{A,B\} \geq 0$$

where

$$A: = \max_{\substack{J\subseteq H\\ J\neq\emptyset}} \frac{[\sum_{i\in H} \sum_{j\in J} p_i p_j a_i (b_i a_j - a_i b_j)]^2}{P_J \sum_{i\in H} p_i a_i^2 - (\sum_{i\in J} p_i a_i)^2}$$

and

$$B: = \max_{\substack{J\subseteq H\\ J\neq\emptyset}} \frac{[\sum_{i\in H} \sum_{j\in J} p_i p_j b_i (b_i a_j - a_i b_j)]^2}{P_J \sum_{i\in H} p_i b_i^2 - (\sum_{i\in J} p_i b_i)^2}$$

and $P_J = \sum_{j\in J} p_j$.

Proof. Let J be a part of H. Define the mapping $f_J\colon R \to R$ given by

$$f_J(t) := \sum_{i\in H} p_i a_i^2 [\sum_{i\in H\setminus J} p_i b_i^2 + \sum_{i\in J} p_i (b_i+t)^2] -$$

$$-[\sum_{i\in H\setminus J} p_i a_i b_i + \sum_{i\in J} p_i a_i (b_i+t)]^2.$$

By Cauchy-Buniakowski-Schwarz's inequality is obvious that:

$$f_J(t) \geq 0 \text{ for all } t\in R.$$

On the other hand we have:

$$f_J(t) = \sum_{i\in H} p_i a_i^2 [\sum_{i\in H} p_i b_i^2 + 2t \sum_{i\in J} p_i b_i + t^2 P_J] -$$

$$- [\sum_{i\in H} p_i a_i b_i + t \sum_{i\in J} p_i a_i]^2 = t^2 [P_J \sum_{i\in H} p_i a_i^2 - (\sum_{i\in J} p_i a_i)^2] +$$

$$+ 2t[\sum_{i\in H} p_i a_i^2 \sum_{i\in J} p_i b_i - \sum_{i\in H} p_i a_i b_i \sum_{i\in J} p_i a_i] +$$

$$+ [\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i a_i b_i)^2]$$

for all $t\in R$.

Since

$$P_J \sum_{i\in H} p_i a_i^2 - (\sum_{i\in J} p_i a_i)^2 \geq P_J \sum_{i\in J} p_i a_i^2 - (\sum_{i\in J} p_i a_i)^2 > 0$$

because $a_i \neq a_j$ for all i,j with $i\neq j$, thus by the inequality $f_J(t) \geq 0$ for all $t\in R$ we get:

$$0 \leq \frac{1}{4}\Delta = [\sum_{i\in H} \sum_{j\in H} p_i p_j a_i (a_i b_j - a_j b_i)]^2 -$$

$$- [P_J \sum_{i\in H} p_i a_i^2 - (\sum_{i\in J} p_i a_i)^2][\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i a_i b_i)^2]$$

from where results the inequality:

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i a_i b_i)^2 \geq A.$$

The second part goes likemise for the mapping $g_J: R \to R$ given by

$$g_J(t) := [\sum_{i\in H\setminus J} p_i a_i^2 + \sum_{i\in J} p_i (a_i+t)^2] \sum_{i\in H} p_i b_i^2 -$$

$$- [\sum_{i\in H\setminus J} p_i a_i b_i + \sum_{i\in J} p_i b_i (a_i+t)]^2$$

and we will omit the details.

The following corollaries also holds.

<u>Corollary 1</u>. In the above assumptions we have:

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i a_i b_i)^2 \geq \tag{2}$$

$$\geq \max\{\frac{[\sum_{i,j\in H} p_i p_j a_i (b_i a_j - b_j a_i)]^2}{P_H \sum_{i\in H} p_i a_i^2 - (\sum_{i\in H} p_i a_i)^2}, \frac{[\sum_{i,j\in H} p_i p_j b_i (b_i a_j - b_j a_i)]^2}{P_H \sum_{i\in H} p_i b_i^2 - (\sum_{i\in H} p_i b_i)^2}\} \geq ($$

The proof is obvious by the above theorem choosing $J=H$.

Corollary 2. In the above assumptions we have:

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \tag{3}$$

$$\geq \frac{1}{\operatorname{card}(H)-1}\max\left\{\frac{\sum_{j\in H} p_j\left(\sum_{i\in H} p_i a_i (b_i a_j - a_j b_i)\right)^2}{\sum_{i\in H} p_i a_i^2}, \frac{\sum_{j\in H} p_j\left(\sum_{i\in H} p_i b_i (b_i a_j - b_j a_i)\right)^2}{\sum_{i\in H} p_i b_i^2}\right\} \geq 0.$$

Proof. Choosing in the above theorem $J = \{j\}$ we get the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \frac{\left[p_j \sum_{i\in H} p_i a_i (b_i a_j - a_i b_j)\right]^2}{p_j \sum_{i\in H} p_i a_i^2 - p_j^2 a_j^2}$$

from where we obtain

$$\left(\sum_{i\in H} p_i a_i^2 - p_j^2 a_j^2\right)\left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq$$

$$\geq p_j\left[\sum_{i\in H} p_i a_i (b_i a_j - a_i b_j)\right]^2 \quad \text{for all } j\in H.$$

Now, summing these inequalities over $j\in H$ we get:

$$(\operatorname{card}(H)-1)\sum_{i\in H} p_i a_i^2\left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq$$

$$\geq \sum_{j\in H} p_j\left[\sum_{i\in H} p_i a_i (b_i a_j - a_i b_j)\right]^2$$

from where we get the first part of (3).

The second part goes likemise and we omit the details.

Remark. Suppose that $(a_i)_{i\in N}$, $(b_i)_{i\in N}$ be are real numbers with $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ $(i,j\in N)$. Then for all $n\in N$, $n \geq 2$, one has the inequalities:

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq$$

$$\geq \max\left\{\frac{\left[\sum_{i,j=1}^{n} a_i (a_i b_j - a_j b_i)\right]^2}{n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2}, \frac{\left[\sum_{i,j=1}^{n} b_i (a_i b_j - a_j b_i)\right]^2}{n\sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} b_i\right)^2}\right\} \geq 0$$

and

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq$$

$$\geq \frac{1}{n-1}\max\left\{\frac{\sum_{i=1}^{n}\left(\sum_{i=1}^{n} a_i(b_i a_j - a_j b_i)\right)^2}{\sum_{i=1}^{n} a_i^2}, \frac{\sum_{i=1}^{n}\left(\sum_{i=1}^{n} b_i(b_i a_j - a_j b_i)\right)^2}{\sum_{i=1}^{n} b_i^2}\right\} \geq 0.$$

For other refinements or connected results with the classical inequality of Cauchy-Buniakowski-Schwarz we send to [1-14] where further references are given.

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НЕРАВЕНСТВО ЗА РЕАЛНИ БРОЕВИ НА КОШИ-БУЊАКОВСКИ-ШВАРЦ

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Р е з и м е

Даден е нов приод на добро познатото неравенство на Коши-Буњаковски-Шварц за реални броеви.

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