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UNITARY EQUIVALENCE OF UNILATERAL OPERATOR VALUED WEIGHTED SHIFTS WITH QUASI-INVERTIBLE WEIGHTS

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Abstract. In this paper we consider the problem of unitary equivalence of unilateral operator valued weighted shifts with quasi-invertible weights.

Let $\mathcal{H}$ be a complex Hilbert space, let (x,y) be the scalar product of vectors x and y in $\mathcal{H}$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all (bound linear) operators from $\mathcal{H}$ to $\mathcal{H}$.

Let $(A_n)_{n=0}^{\infty}$ be an uniformly bounded sequence of bounded and linear operators on $\mathcal{H}$.

$\ell^2(\mathcal{H}) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}$, for all n is a notation for the space of infinite seauences of vectors $(f_k)_{k=0}^{\infty}$, $f_k \in \mathcal{H}$ such that $||f||^2 = \sum_{k=0}^{\infty} ||f_k||^2 < \infty$ with a scalar product defined by $(f,g) = \sum_{k=0}^{\infty} (f_k, g_k)$.

The operator on $\ell^2(\mathcal{H})$ given by

$$A(f_0,f_1,f_2,\ldots) = (0,A_0f_0,A_1f_1,A_2f_2,\ldots)$$

is called the unilateral operator valued weighted shift with weights $(A_n)_{n=0}^{\infty}$.

Two operators A and B are called unitary equivalence if there exists an unitary operator U on $\mathcal{H}$ such that $AU = UB$.

A bounded operator A is quasi-invertible if A is injective and has a dense range (i.e. $\mathrm{Ker}A=\mathrm{Ker}A^* = \{0\}$).

Throughout this paper we consider operator valued weighted shifts with quasi-invertible weights and we shall call them operator valued weighted shifts.

Lemma. Let A and B be unilateral operator valued weighted shifts with weights $(A_k)_{k=0}^{\infty}$ and $(B_k)_{k=0}^{\infty}$ respectively and

$X \in \mathcal{B}(\ell^2(\mathcal{H}))$ with matrix $(X_{ij})_{i,j=0}^{\infty}$, then AX=XB if and only if $X_{ij}=0$, $j \geq i+1$ and $A_i X_{ij} = X_{i+1,j+1} B_j$, $i,j=0,1,2,\ldots$

Proof. Let

$$A = \begin{bmatrix} 0 & 0 & . \\ A_0 & 0 & . \\ 0 & A_1 & . \\ . & . & . \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & . \\ B_0 & 0 & . \\ 0 & B_1 & . \\ . & . & . \end{bmatrix} \qquad X = \begin{bmatrix} X_{00} & X_{01} & X_{02} & . \\ X_{10} & X_{11} & X_{12} & . \\ X_{20} & X_{21} & X_{22} & . \\ . & . & . & . \end{bmatrix}$$

If AX=XB, then $X_{0j}B_{j-1}=0$ for $j \geq 1$.

Since B_j has a dense range in $\mathcal{H}$ for all j then $X_{0j}=0$, $j \geq 1$. Also, $A_i X_{ij} = X_{i+1,j+1} B_j$ for all i and j. If we suppose that $X_{nj}=0$ for $j \geq n+1$, then $X_{n+1,j+1}B_j=0$. Since B_j has dense range in $\mathcal{H}$ it follows that $X_{n+1,j+1}=0$ for $j \geq n+1$. Conversly, let

$$X = \begin{bmatrix} X_{00} & 0 & 0 & . \\ X_{10} & X_{11} & 0 & . \\ X_{20} & X_{21} & X_{22} & . \\ . & . & . & . \end{bmatrix}$$

and $A_i X_{ij} = X_{i+1,j+1} B_j$ for every i and j. Then

$$AX = \begin{bmatrix} 0 & 0 & 0 & . \\ A_0 X_{00} & 0 & 0 & . \\ A_1 X_{10} & A_1 X_{11} & 0 & . \\ . & . & . & . \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . \\ X_{11} B_0 & 0 & 0 & . \\ X_{21} B_1 & X_{22} B_2 & 0 & . \\ . & . & . & . \end{bmatrix} = XB \quad \square$$

Theorem 1. The unilateral operator valued weighted shifts A and B are unitarly equivalent if and only if there is an unitary operator U_0 on $\mathcal{H}$ such that

$$\|A_{k-1} \ldots A_0 U_0 x\| = \|B_{k-1} \ldots B_0 x\| \text{ for all } k \in \mathbb{N} \text{ and } x \in \mathcal{H} \qquad (*)$$

Proof. Let us assume that the operators A and B are unitary equivalence. Then there exists an unitary operator U on $\ell^2(\mathcal{H})$ such that AU=UB. According to the Lemma and the fact that U*A=BU* we have that the operator U is diagonal, $U=\overset{\infty}{\Sigma}\oplus U_n$ and each U_n is unitary on $\mathcal{H}$. Also, $A_i U_i = U_{i+1} B_i$ for all $i \geq 0$. Let $k > 0$ and $x \in \mathcal{H}$, then

$$\|A_{k-1} \ldots A_0 U_0 x\| = \|A_{k-1} \ldots U_1 B_0 x\| =$$
$$= \|U_k B_{k-1} \ldots B_0 x\| = \|B_{k-1} \ldots B_0 x\|$$

Conversly, suppose U_0 is unitary on $\mathbb{H}$ and (*) is hold. Now, we are going to define the operators U_i.

Let $U_1: B_0\mathbb{H} \to A_0\mathbb{H}$ is such that $U_1 y = A_0U_0x$ for all $y \in B_0\mathbb{H}$, and $x \in \mathbb{H}$ is such that $B_0x = y$. We shall prove that U_1 is an isometry and that $U_1(B_0\mathbb{H}) = A_0\mathbb{H}$.

$||U_1y|| = ||A_0U_0x|| \overset{(*)}{=} ||B_0x|| = ||y||$, for all $y \in B_0\mathbb{H}$.

Let $x \in A_0\mathbb{H}$ and $x_1 \in \mathbb{H}$ is such that $A_0U_0x_1 = x$, and let $y = B_0x_1$. Then $y \in B_0\mathbb{H}$ and $U_1y = A_0U_0x_1 = x$.

Let U_1 is the minimal unitary extension of U_1 on $\mathbb{H}$ $(\mathbb{H} = \overline{B_0\mathbb{H}})$. So U_1 is unitary operator on $\mathbb{H}$, and $U_1B_0 = A_0U_0$.

Let $U_2, U_3, \dots, U_{n-1}$ are unitary operators on $\mathbb{H}$ such that $U_iB_{i-1} = A_{i-1}U_{i-1}$, $i \in \{1,2,\dots,n-1\}$. We shall define the operator U_n in the following way:

Let $U_n: B_{n-1}\dots B_0\mathbb{H} \to A_{n-1}\dots A_0\mathbb{H}$ is such that $U_ny = A_{n-1}\dots A_0U_0x$ for all $y \in B_{n-1}\dots B_0\mathbb{H}$, and $x \in \mathbb{H}$ is such that $B_{n-1}\dots B_0x = y$. We shall prove that U_n is an isometry, $U_n(B_{n-1}\dots B_0\mathbb{H}) = A_{n-1}\dots A_0\mathbb{H}$ and $U_nB_{n-1} = A_{n-1}U_{n-1}$.

(i) U_n is an isometry: Let $y \in B_{n-1}\dots B_0\mathbb{H}$ then

$||U_ny|| = ||A_{n-1}\dots A_0U_0x|| \overset{(*)}{=} ||B_{n-1}\dots B_0x|| = ||y||$

(ii) $U_n(B_{n-1}\dots B_0\mathbb{H}) = A_{n-1}\dots A_0\mathbb{H}$

Let $x \in A_{n-1}\dots A_0\mathbb{H}$, $x_1 \in \mathbb{H}$ is such that $x = A_{n-1}\dots A_0U_0x_1$ and $y = B_{n-1}\dots B_0x_1$. Then $y \in B_{n-1}\dots B_0\mathbb{H}$ and $A_ny = A_{n-1}\dots A_0U_0x_1 = x$.

$$
\begin{aligned}
\text{(iii)}\quad & (U_nB_{n-1} - A_{n-1}U_{n-1})B_{n-2} \dots B_0 = \\
& U_nB_{n-1}B_{n-2}\dots B_0 - A_{n-1}U_{n-1}B_{n-2} \dots B_0 = \\
& U_nB_{n-1}B_{n-2}\dots B_0 - A_{n-1}A_{n-2}U_{n-2}B_{n-3} \dots B_0 = \\
& \quad \dots \qquad \dots \\
& U_nB_{n-1}B_{n-2}\dots B_0 - A_{n-1}A_{n-2}\dots A_0U_0 = 0
\end{aligned}
$$

Since $B_{n-2}\dots B_0$ has a dense range in $\mathbb{H}$, then $U_nB_{n-1} = A_{n-1}U_{n-1}$.

Using (i) and (ii) U_n can be extended to unitary operator on $\mathbb{H}$ (we shall call it U_n).

In that way we get a sequence $(U_n)_{n=0}^{\infty}$ of unitary operators on $\mathbb{H}$ such that $U_nB_{n-1} = A_{n-1}U_{n-1}$ for all $n \in \mathbb{N}$.

Let $U=\Sigma\oplus U_n$ and $f\in\ell^2(\mathbb{H})$. Then

$$||Uf||^2 = \Sigma||U_nf_n||^2 = \Sigma||f_n||^2 = ||f||^2$$

also $UU^* = \Sigma\oplus U_n^*U_n = I$.

So U is an unitary operator from

$$\ell^2(\mathbb{H}) \text{ to } \ell^2(\mathbb{H}) \text{ and } AU=UB. \ \square$$

Corollary. Two unilateral operator valued weighted shifts A and B are unitarily equivalent if and only if the operators

$$(A_k\ldots A_o)^*(A_k\ldots A_o) \text{ and } (B_k\ldots B_o)^*(B_k\ldots B_o)$$

are unitarily equivalent for all $k\in\mathbb{N}$.

Proof. Using Theorem 1 it is easy to see that A and B are unitarily equivalent if and only if there exists an unitary operator U_o on $\mathbb{H}$ such that

$$(x,U_o^*(A_k\ldots A_o)=(A_k\ldots A_o)U_ox) =$$
$$(x,(B_k\ldots B_o)^*(B_k\ldots B_o)x)$$

for all $x\in\mathbb{H}$ and $k\in\mathbb{N}$. $\square$

Theorem 2. The unilateral operator valued weighted shift A with quasi-invertible weights is unitarily equivalent to an unilateral operator valued weighted shift with quasi-invertible and positive weights.

Proof. Let $A_k=U_kP_k$ be the polar decomposition of A_k. Then U_k is unitary operator

(U_k is an isometry on $(\mathrm{Ker}U_k)\perp = (\mathrm{Ker}P_k)\perp =$
$(\mathrm{Ker}A_k)\perp = \mathbb{H}$ and $\overline{U_k\mathbb{H}} = \mathbb{H}$) and P_k is positive

Let $P=\Sigma\oplus P_k$, $U=\Sigma\oplus U_k$ and S is an unilateral shift operator of multiplicity $\dim(\mathbb{H})$. $P\in\mathcal{B}(\ell^2(\mathbb{H}))$ since

$$||P|| = \sup_k||P_k|| = \sup_k||A_k|| = ||A||$$

Then the operators A and SUP are unitarily equivalent. Also the operators S and SU are unitarily equivalent. (Let $V=\Sigma\oplus V_k$ such that $V_o=I$,

$$V_k = S_{k-1}U_{k-1}S_{k-2}U_{k-2}\ldots S_oU_oS_o^*S_1^*\ldots S_{k-1}^*)$$

Then $A=SUP=VSV^*P=V(SV^*PV)V^*$. Thus the operators A and SV^*PV are unitarily equivalent. Now V^*PV is a diagonal operator with positive quasi-invertible operators on its diagonal and so SV^*PV is an unilateral operator valued weighted shift with positive quasi-invertible weights. □

Theorem 3. If A and B are unilateral operator valued weighted shifts with quasi-invertible weights on $\mathcal{H}$ and if there exists a quasi-invertible operator S on $\ell^2(\mathcal{H})$ such that $AS=SB$ and $A^*S=SB^*$ then A and B are unitarily equivalent.

Proof. According to the Lemma we have that operator S has lower triangular matrix. From $A^*S=SB^*$ we have that $A_i^*S_{i+1,o}=0$, $i=0,1,2,\ldots$. This means that $S_{i+1,o}\mathcal{H}\subseteq \operatorname{Ker}A_i^*=\{0\}$, so $S_{i,o}=0$ for all $i > 0$. Also $A_i^*S_{i+1,j+1}=S_{i,j}A_j^*=0$ for $j\neq i$, so we have that $S_{i,j}=0$ for $i\neq j$, i.e. S is diagonal operator. Let $S=UP$ be the polar decomposition of S. U is unitary operator and $P=(S^*S)^{1/2}$. Then, $BP^2=B(S^*S)=(SB^*)^*S=(A^*S)^*S=S^*AS=S^*SB=P^2B$. Since P is uniform limit of the sequence $(P_k(P^2))$ where (P_k) is a sequence of polynomials it follows that $BP=PB$. Thus by $AUP=UPB=UBP$ we have

$$\mathcal{H}=\overline{P\mathcal{H}}\subseteq \operatorname{Ker}(AU-UB) \text{ i.e. } AU=UB. \ \square$$

Corollary. If the unilateral operator valued weighted shifts A and B are quasisimilar (there exist quasi-invertible operation on $\ell^2(\mathcal{H})$ S and T such that $AS=SB$ and $BT=TA$) and if $S^*=T$ then A and B are unitarily equivalent.

R E F E R E N C E S

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УНИТАРНА ЕКВИВАЛЕНЦИЈА НА УНИЛАТЕРАЛНИ ОПЕРАТОРСКО ТЕЖИНСКИ ШИФТОВИ СО КВАЗИ-ИНВЕРЗИБИЛНИ ТЕЖИНИ

Марија Оровчанец

Р е з и м е

Во овој труд се разгледува проблемот на унитарна еквиваленција на едностраните операторско тежински шифтовии чии тежини се квази-инверзибилни (еден-еден и со густ ранг) оператори.

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