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SEMI-ORTHOGONAL PROPERTY OF A CLASS OF THE GAUSS' HYPERGEOMETRIC POLYNOMIALS

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Abstract. We establish a semi-orthogonal property for a class of Gauss' hypergeometric polynomials with an elementary weight function and employ it to generate a theory concerning the finite series expansion involving the Gauss' hypergeometric polynomials.

Key Words. Gauss' Hypergeometric polynomials, Semi-Orthogonal property, Saalschutz's theorem, Fox's H-function.

AMS (MOS) Subject classification 33C25, 33C40.

1. Introduction. Gauss' hypergeometric polynomials $x^n\ {}_2F_1(-n,b;c;\frac{1}{x})$ lead to the generalization of the classical polynomials associated with the names of Jacobi, Gegenbauer, Legendre and Chebychev. Therefore, they appear to represent a very important class of polynomials. In view of this, it seems worthwhile to investigate the matter of their orthogonality and other important aspects.

In this paper, we evaluate an integral involving the Gauss' hypergeometric polynomials and employ it to establish a semi-orthogonal property of the Gauss' hypergeometric polynomials. We further apply this property to develop a theory regarding the finite series expansion involving the Gauss' hypergeometric polynomials. We also employ the integral to evaluate an integral for Fox's H-function [4]. We also present some particular cases of our integral and semi-orthogonal property, one of which is known.

The Jacobi polynomials constitute an important, and a rather wide class of hypergeometric polynomials, from which Chebyshev, Legendre, and Gegenbauer polynomials follow as special cases. Their orthogonality property, with the non-negative weight function $(1-x)^a(1+x)^b$ on the interval $[-1,1]$, for $a > -1$, $b < -1$, is usually derived by the use of the associated differential equation and Rodrigues' formula. In this paper, we introduce a direct method of proof, which is much simpler and elegant to establish orthogonality properties of the hypergeometric polynomials. Our method could be employed to establish the orthogonalities of the Jacobi polynomials and other related hypergeometric polynomials.

The class of Gauss' hypergeometric polynomial is defined and represented as follows:

$$F_n^{(b,c)}(x) = x^n {}_2F_1(-n,b;c;\tfrac{1}{x}) = \sum_{r=0}^{n} \frac{(-n)_r (b)_r}{(c)_r r!}(x)^{n-r}, \quad n=0,1,2,\ldots \tag{1.1}$$

provided c is not zero or a negative integer.

The following integral is required in the proofs:

$$\int_0^1 x^h (1-x)^{b-c-n} F_n^{(b,c)}(x)\,dx = = (-1)^n \frac{(1+h+b)_n \Gamma(1+h)\Gamma(1+b-c)}{(c)_n \Gamma(2+b+h-c)}, \quad n=0,1,2,\ldots \tag{1.2}$$

where $\mathrm{Re}\,h > -1$, $\mathrm{Re}(h+b) > -1-n$, $\mathrm{Re}(b-c) > -1$.

<u>Proof</u>. The integral (1.2) is established by expressing the hypergeometric polynomial in the integrand as its series representation (1.1), interchanging the order of integration and summation, evaluating the resulting integral with the help of the Beta integral [2,p.9]:

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0,\ q > 0 \tag{1.3}$$

and simplifying with the help of the following form of the formula [2,p.3,(4)]:

$$\Gamma(1+a-n) = \frac{(-1)^n \Gamma(1+a)}{(-a)_n}, \tag{1.4}$$

we get

$$= \frac{\Gamma(h+n+1)\Gamma(b-c-n+1)}{\Gamma(2+h+b-c)} {}_3F_2\left[\begin{matrix}-n, b, c-b-h-1; 1\\ c, -h-n\end{matrix}\right] \tag{1.5}$$

It can easily be verified that the generalized hypergeometric series (1.5) is Saalschutzian. Therefore, applying the Saalschutz's theorem [2,p.188,(3)]:

$$ {}_3F_2\left[\begin{matrix}-n,a,b;1\\ c,1-c+a+b-n\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \tag{1.6} $$

and simplifying the right hand side of (1.2) is obtained.

2. Semi-orthogonal property. The semi-orthogonal property to be established is

$$ \int_0^1 x^{-1-b-m}(1-x)^{b-c-n} F_m^{(b,c)}(x) F_n^{(b,c)}(x)\,dx \tag{2.1} $$

$$ = 0, \text{ if } m < n \tag{2.1a} $$

$$ = (-1)^n \frac{(b)_n n!\,\Gamma(-b)\Gamma(1+b-c)}{(c)_n(1+b)_n\Gamma(1-c)}, \text{ if } m = n \tag{2.1b} $$

where Re(b-c) > -1, (Reb < -m, Reb > -n ➡ when m=n, Reb ≠ -n).

Proof. To prove (2.1), we write its left hand side in the form

$$ \sum_{r=0}^{m} \frac{(-m)_r(b)_r}{r!(c)_r} \int_0^1 x^{-1-b-r}(1-x)^{b-c-n} F_n^{(a,b)}(x)\,dx \tag{2.2} $$

On evaluating the last integral in (2.2) by using the integral (1.2), we get

$$ \sum_{r=0}^{m} \frac{(-m)_r(b)_r}{r!(c)_r}(-1)^n \frac{\Gamma(-b-r)\Gamma(1+b-c)(-r)_n}{(c)_n\Gamma(1-r-c)} \tag{2.3} $$

If r < n, the numerator of (2.3) vanishes, and since r runs from 0 to m, it follows that (2.3) also vanishes, when m < n. Therefore, it is clear that for m < n all terms of (2.3) vanish, which proves (2.1a).

When m=n, using the standard result $(-n)_n = (-1)^n n!$, and simplifying, the right hand side of (2.1b) follows from (2.3).

Note: The integral (2.1) exists for m=n+1,n+2,n+3,..., and yields a series of interesting results.

3. Particular cases of the integral

(i) In (1.2), putting $\frac{1}{x} = \frac{1+y}{2}$ and using (1.1), it reduces to the form

$$ \int_1^\infty (y+1)^{c-h-b-2}(y-1)^{b-c-n} F_n^{(b,c)}\left(\frac{2}{1+y}\right)dy $$
$$ = (-1)^n \frac{(1+h+b)_n\Gamma(1+h)\Gamma(1+b-c)}{2^{h+n+1}(c)_n\Gamma(2+b+h-c)} \tag{3.1} $$

where the conditions are same as in (1.2).

(ii) In (3.1), setting n+a+b+1 for b and 1+b for c and using the relation [3,p.170,(16)]:

$$P_n^{(a,b)}(x) = (-1)^n \frac{(1+b)_n}{n!} {}_2F_1\left({-n,n+a+b+1;\atop 1+b}1+x/2\right) \tag{3.2}$$

we get the following integral involving Jacobi polynomials:

$$\int_1^\infty (y+1)^{-a-h-n-2}(y-1)^a P_n^{(a,b)}(y)\,dy = \frac{(2+h+a+b+n)_n \Gamma(1+h)\Gamma(a+n+1)}{n!\,2^{h+n+1}\Gamma(2+h+n+a)}, \tag{3.3}$$

where Reh > -1, Re(h+a+b) > -2-2n, Rea > -1.

4. Particular cases of the semi-orthogonal property

(i) In (3.1), putting $\frac{1}{x} = \frac{1+y}{2}$ and using (1.1), we obtain

$$\int_1^\infty (y+1)^{c-1}(y-1)^{b-c-n} F_m^{(b,c)}\left(\frac{2}{1+y}\right) F_n^{(b,c)}\left(\frac{2}{1+y}\right) dy \tag{4.1}$$

$$= 0, \text{ if } m < n \tag{4.1a}$$

$$= (-1)^n \frac{2^{b-n} n!\,(b)_n \Gamma(-b)\Gamma(1+b-c)}{(c)_n (1+b)_n \Gamma(1-c)}, \text{ if } m = n \tag{4.1b}$$

where Re(b-c) > n-1.

(ii) In (4.1) putting m = n, setting n+a+b+1 for b and 1+b for c and using the relation (3.2), we get a known result [1,p.3,(2.1)]:

$$\int_1^\infty (y-1)^a (y+1)^b [P_n^{(a,b)}(y)]^2 dy = \frac{2^{a+b+1}\Gamma(1+a+n)\Gamma(1+b+n)\Gamma(-1-a-b)\Gamma(2+a+b)}{n!\,(1+a+b+2n)\Gamma(1+a+b+n)\Gamma(-b)\Gamma(1+b)} \tag{4.2}$$

where Rea > -1.

5. Finite series expansion involving hypergeometric polynomials. Based on the relations (2.1a) and (2.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a finite series expansion of the Gauss' hypergeometric polynomials. Specially if F(x) is a suitable function defined for all x, we consider for expansion of the general form

$$F(x) = \sum_{m=0}^{n} A_m \, x^{-m} F_m^{(b,c)}(x), \quad 0 < x < 1, m \leq n \tag{5.1}$$

where the expansion coefficients A_m are given by

$$A_m = (-1)^m \frac{(c)_m (1+b)_m \Gamma(1-c)}{(b)_m m! \, (-b) \, (1+b-c)} \int_0^1 F(x) x^{-1-b} (1-x)^{b-c-m} F_m^{(b,c)}(x) dx \tag{5.2}$$

where $Re(b-c) > -1$, $Re b \neq -m$.

6. Integral involving Fox's H-function and Gauss' hypergeometric polynomial. The integral to be established is

$$\int_0^1 x^h (1-x)^{b-c-n} F_n^{(b,c)}(x) H_{p,q}^{u,v}\left[zx^k \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \frac{(-1)^n \Gamma(1+b-c)}{(c)_n} H_{p+2,q+2}^{u,v+2}\left[z \middle| \begin{matrix} (-h,k), (-h-b-n,k), (a_p,e_p) \\ (b_q,f_q), (-h-b,k), (c-b-h-1,k) \end{matrix} \right] \tag{6.1}$$

where k is a positive number, and

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \geq 0, \quad \sum_{j=1}^{v} e_j - \sum_{j=v+1}^{p} e_j + \sum_{j=1}^{u} f_j - \sum_{j=u+1}^{p} f_j = B > 0, \ |\arg z| < \tfrac{1}{2}B,$$

$Re(b-c)+k \min\limits_{1\leq j\leq u} [Re b_j/f_j] > -1$, $-Re h - k \max\limits_{1\leq j\leq v} [Re(a_j-1)/e_j] < 1$,

$-Re(h+b)-k \max\limits_{1\leq j\leq v} [Re(a_j-1)/e_j] < 1+n$, and (a_p, e_p) represents the set of parameters $(a_1,e_1),\ldots,(a_p,e_p)$.

Proof. The integral (6.1) is obtained by expressing the H-function in the integrand as a Mellin-Barnes type integral [5,p.2,(1.1.1)], changin the order of integrations and evaluating the inner-integral with the help of (1.2), and using the standard method to evaluate such integrals [5].

Note: Since on specializing the parameters, the H-function may be reduced to almost all special functions appearing in pure and applied mathematics [5,p.144-159]. Therefore the integral (6.1) is of a very general character and hence may encompass several cases of interest. Our result (6.1) is a master or key formula from which a large number of results can be derived for Meijer's G-function, MacRobert's E-function, Hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, orthogonal polynomials, trigonometric functions and other related functions.

R E F E R E N C E S

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ПОЛУОРТОГОНАЛНО СВОЈСТВО НА ЕДНА КЛАСА GAUSS-ОВИТЕ ХИПЕРГЕОМЕТРИСКИ ПОЛИНОМИ

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Р е з и м е

Во работата воспоставено е полуортогонално својство за една класа од Gauss-ови хипергеометриски полиноми со една елементарна тежинска функција, што е искористена за градење теорија за развој на Gauss-овите хипергеометриски полиноми во конечни суми.

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