k — SEMINETS

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This paper introduces the concept of k-seminets, $k \geqslant 3$, establishes some of its basic properties, establishes the characterization of 3-seminets by means of a class of partial quasigroups and establishes the characterization of k-seminets, $k > 3$, by means of a class of orthogonal systems of partial quasigroups.

k-Seminet, basides, qeneralises the k-net [1], whereas 3-seminets and 3-nets are special cases of halfnets of V. Havel [2].

The elements of the non-empty est τ will be said to be the *points*, and the elements of the sets $L_i \subseteq P(\tau) \setminus \{\varnothing\}$, $i \in \{1, \ldots, k\}$, $k \in N \setminus \{1,2\}$, to be the *lines*. Let, further, $L_i \cap L_j = \varnothing$ each $i \in \{1, \ldots, k\}$ and each $j \in \{1, \ldots, k\}$, if $i \neq j$. $(\tau, L_1, \ldots, L_k)$ will be said to be a k-seminet iff the following conditions are fulfilled:

R1. $(\forall l_i)(\forall l_j)(\forall i \in \{1, \ldots, k\})(\forall j \in \{1, \ldots, k\})[(l_i \in L_i \wedge l_j \in L_j \wedge$

$\wedge\, i \neq j) \Rightarrow \mu(l_i \cap l_j)'') \leqslant 1]$; i

R2. $(\forall T \in \tau)(\exists! l_1 \in L_1) \ldots (\exists! l_k \in L_k)(T \in l_1 \wedge \ldots \wedge T \in l_k)$.

If in R1. $\mu(l_i \cap l_j) = 1$, is a k-seminet $(\tau, L_1, \ldots, L_k)$ is a k-net.

Figures 1 and 2 represent, in turn, a 3-seminet and a 4-seminet respectively.

If τ is a finite set, the k-seminet will be said to be finite. We shall primarily be interested in the finite k-seminets.

$m \in N$ is an *L-order* of the k-seminet $(\tau, L_1, \ldots, L_k) \overset{\text{def}}{\Longleftrightarrow} m = \text{Max}$ $\{\mu L_i \mid i \in \{1, \ldots, k\}\}$. $n \in N$ is a *T-order* of the k-seminet $\overset{\text{def}}{\Longleftrightarrow} n = \text{Max}\ \{\mu\, l \mid l \in L_1 \cup \ldots \cup L_k\}$.

'' Cardinal number of the set $l_i \cap l_j$.

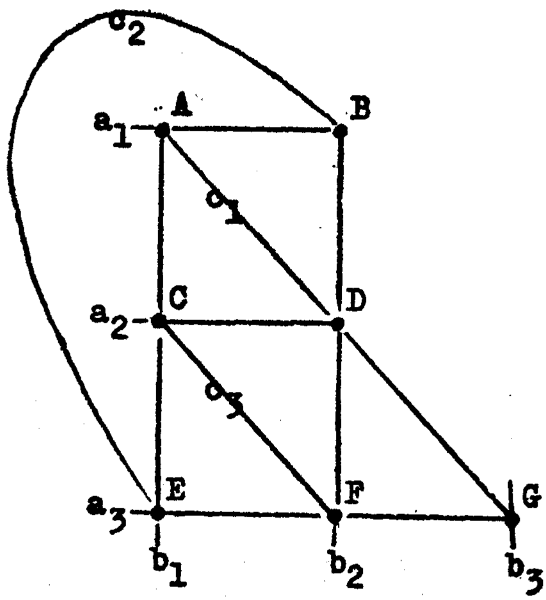

Fig.1.

$L_a = \{a_1, a_2, a_3\}$

$L_b = \{b_1, b_2, b_3\}$

$L_c = \{c_1, c_2, c_3\}$

$a_1 = \{A, B\}, a_2 = \{C, D\}, a_3 = \{E, F, G\}$

$b_1 = \{A, C, E\}, b_2 = \{B, D, F\}, b_3 = \{G\}$

$c_1 = \{A, D, G\}, c_2 = \{B, E\}, c_3 = \{C, F\}$

The concepts, L-order and T-order generalize the concept of the order of the k-net.

THEOREM 1. Let $n \in N$ be a T-order of the k-seminet $(\tau, L_1, \ldots, L_k)$ $k \in N \setminus \{1, 2\}$. Then

$$k \leqslant n + 1 + \Delta,$$

where $\Delta = \min \{T_l - D_l \mid T \in \tau \wedge l \in L_1 \cup \ldots \cup L_k \wedge T \notin l\}$, $k - 1 - T_l$ the number of the lines passing through the point T and cutting the line $l \not\ni T$, and $n - D_l$ is the total number of the points upon the line l.

Let $Q \neq \emptyset$, $D \neq \emptyset$ and $D \subseteq Q \times Q$. If $A : D \to Q$, then (Q, A) is said to be a partial groupoid. If in the partial groupoid (Q,A) each pair of elements in the equation $A(x, y) = z$ uniquely defines the third one, if such exists, (Q, A) is said to be a partial quasigroup. The k-seminets will be defined by means of *regular partial quasigroups*: the partial quasigroup (Q,A) will be said to be regular, ift it satisfies the following conditions:

$$1°.\ (\forall (i,j))\,((i,j) \in D \Rightarrow [(\exists j')(j \neq j' \wedge (i,j') \in D)] \vee [(\exists i')(i \neq i' \wedge (i',j) \in D]); \text{ and}$$

$$2°.\ (\forall (i,j))\,[A(i,j) = t \Rightarrow [\{(i,j)\} = (\{i\} \times Q) \cap D] \vee \{(i,j)\} = (Q \times \{j\}) \cap D] \Rightarrow (\exists (i',j'))\,((i,j) \neq (i',j') \wedge A(i',j') = t))].$$

THEOREM 3. To each 3-seminet there corresponds a regular partial quasigroup, and vice versa.

Proof

Let (Q,A) be a regular partial quasigroup. The elements of the set $\tau = \{(i, j, t) / A(i, j) = t\}$ will be said to be points, and each of the non-empty sets $a_i = \{(i, j, t) \mid A(i,j) = t\}$, $b_j = \{(i, j, t) \mid A(i,j) = t\}$, $c_t = \{(i, j, t) / A(i,j) = t\}$, will be said to be lines. (τ, L_a, L_b, L_c) is a 3-seminet, if $L_a = \{a_i \setminus a_i \neq \emptyset\}$, $L_b = \{b_j \mid b_j \neq \emptyset\}$ and $L_c = \{c_t \mid c_t \neq \emptyset\}$.

Let, now, (τ, L_a, L_b, L_c) be a 3-seminet, Q be a set of the power of the set L_x, $x \in \{a, b, c\}$, with the highest power, and mappings $f : L_\alpha \to Q$, $g : L_b \to Q$, $h : L_c \to Q$ be injections. If (Q, A) is defined in the following way

$$A(fa_i, gb_j) = hc_t \overset{\text{def}}{\Longleftrightarrow} (a_i \in L_a \wedge b_j \in L_b \wedge a_i \cap b_j \cap c_t = \{T\} \wedge T \in \tau),$$

then (Q, A) is a regular partial quasigroup.

In each quasigroup, the conditions 1° and 2° are satisfied. T3. generalizes the wellknown characterization of 3-nets by means of quasigroups.

Let (Q, A) and (Q, B) be partial groupoids with the same domain $D = \mathcal{D}A = \mathcal{D}B$, $D \subseteq Q \times Q$. A and B are said to be orthogonal iff the system of equations

$$A(x, y) = a, \quad B(x, y) = b$$

is uniquely soluble for each $(a, b) \in Q^2$ for which a solution exists. If we introduce

$$0_{AB}(x, y) \overset{\text{def}}{=} (A(x, y), B(x, y)),$$

then, A and B can be said to be orthogonal iff 0_{AB} is a byjection of the set D upon the set $\mathcal{R}0_{AB}$.

The orthogonal operations A and B will be said to be *regularly orthogonal* iff the following condition is fulfilled:

$$3°. \ (\forall (i,j)) \, [(i,j) \in \mathcal{R}0_{AB} \Rightarrow (\exists j')(j' \neq j \wedge (i,j') \in \mathcal{R}0_{AB}) \vee$$
$$\vee (\exists i')(i' \neq i \wedge (i',j) \in \mathcal{R}0_{AB})].$$

The set of partial operations of the same domian will be said to be an orthogonal system of operations iff each pair of the operations of this set il orthogonal. If each pair of the operations is regularly orthogonal, we shal say that a regular orthogonal system of operations is dealt with.

THEOREM 4. To each k-seminet, $k > 3$, there corresponds a regularly orthogonal system of k—2 regular partial quasigroup, and vice versa.

Proof

Let $A_1, \dots, A_{k-2}$ form a system of regularly ortsogonal regular partial quasigroups. The elements of the set $\tau = \{(e, f, t_1, \dots, t_{k-2}) / A_1(e,f) = t_1 \wedge \dots \wedge A_{k-2}(e,f) = t_{k-2}\}$ will be said to be points, and each of the following non-empty sets will be said to be lines:

$$a_e^{(1)} = \{(e, f, t_1, \dots, t_{k-2}) \,|\, A_1(e,f) = t_1 \wedge \dots \wedge A_{k-2}(e,f) = t_{k-2}\}$$

$$a_f^{(2)} = \{(e, f, t_1, \dots, t_{k-2}) \,|\, A_1(e,f) = t_1 \wedge \dots \wedge A_{k-2}(e,f) = t_{k-2}\}$$

$$\{a_{t_{k-2}}^{(k)} = \{(e, f, t_1, \dots, t_{k-2}) \,|\, A_1(e,f) = t_1 \wedge \dots \wedge A_{k-2}(e,f) = t_{k-2}\}.$$

$(\tau, L_1, \dots, L_k)$ is a k-seminet, if $L_1 = \{a_e^{(1)} \,|\, a_e^{(1)} \neq \emptyset\}, \dots, L_k = \{a_{t_{k-2}}^{(k)} \,|\, a_{t_{k-2}}^{(k)} \neq \emptyset\}$.

Let $(\tau, L_1, \dots, L_k)$ be a k-seminet, $k > 3$. Let Q be a set of the power of the set L_x, $x \in \{1, \dots, k\}$, with the greatest power, and let the mappings

$$f_1 : L_1 \to Q, \dots, f_k : L_k \to Q$$

be injections. Let the operations $A_1, \dots, A_{k-2}$ be defined in the following way:

$$A_i(f_1 l_1, f_2 l_2) = f_{i+2}\, l_{i+2} \overset{\text{def}}{\Longleftrightarrow} l_1 \in L_1 \wedge l_2 \in L_2 \wedge$$
$$l_1 \cap l_2 \cap l_{i+2} = \{T\} \wedge T \in \tau \quad \text{for each} \quad i \in \{1, \dots, k-2\}.$$

The following holds: A_i, $i \in \{1, \ldots, k-2\}$, are regular partial quasigroups; because of R1, they form an orthogonal system, and, beacause the crossings of each of each pair of the sets $L_1, \ldots, L_k$, are empty; they form a regular orthogonal system.

In a non-partial case, each pair of orthogonal operations satisfies the condition 3°. Theorem 4. generalizes the well-known statement about characterization of the k-nets.

Let S_k^n represent a set of all k-seminets of the L-order $n \in N$. If in S_k^n a k-net exists, its τ has the power of n^2. If τ is a set of points of any of k-seminets from S_k^n, then $\mu\tau \leqslant n^2$. Finding the set $M_k^n \subseteq S_k^n$ of k-seminets, the τ' s of which have the gratest power, is considered by the author to be one of the basic tasks of the k-seminet theory.

An affine plane is a k-net in which $k = n + 1$ [1]. Theorem 1. offers a suggestion for the generalization of the affine plane.

The m-dimensional k-seminets are already being studied.

LITERATURE

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