ON THE GROUP PROPERTIES OF SOME CONTRACTIONS

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In this paper we consider some contractions between the tensor spaces $T_2^2 = R^n \otimes R^n \otimes (R^n)^* \otimes (R^n)^*$ and $T_1^1 = R^n \otimes (R^n)^*$.

It is proved that the restriction of a certain contraction on T_1^1

$$T_2^2 \times T_1^1 \to T_1^1$$

induces group properties of the left action induced by this contraction, and at the end we obtain the canonical form of the so obtained Lie group. (In these notations R denotes the real numbers line, and an asterisk is written to the corresponding dual spaces.)

Let T_1^1 and T_2^2 be the tensor spaces

$$T_1^1 = R^n \otimes (R^n)^*,$$

$$T_2^2 = R^n \otimes R^n \otimes (R^n)^* \otimes (R^n)^*$$

and let s_j^i $(1 \leqslant i, j \leqslant n)$ and u_{kl}^{ij} $(1 \leqslant i, j, k, l \leqslant n)$ be the natural coordinate systems in T_1^1 and T_2^2 respectively. The elements of $T_1{}^1$ shall be written by Latin, and those of T_2^2 by Greek letres.

We shall consider the contractions

$$\varphi : T_2^2 \times T_1^1 \to T_1^1$$

and

$$\psi : T_1^1 \times T_2^2 \to T_1^1$$

defined by

$$s_j^i \cdot \varphi(\alpha, a) = u_{jq}^{ip}(\alpha)\, s_p^q(a) \qquad (\alpha, a) \in T_2^2 \times T_1^1, \tag{1}$$

$$s_j^i \cdot \psi(a, \alpha) = s_q^p(a)\, u_{pj}^{qi}(\alpha) \qquad (a, \alpha) \in T_1^1 \times T_2^2, \tag{2}$$

or, if we denote

$$\varphi(\alpha, a) = \alpha\, a \text{ and } \psi(a, \alpha) = a\, \alpha,$$

(1) and (2) can be written as

$$s^i_j(\alpha\ a) = u^{ip}_{jq}(\alpha)\, s^q_p(a), \tag{1'}$$

$$s^i_j(a\ \alpha) = s^p_q(a)\, u^{qi}_{pj}(\alpha). \tag{2'}$$

The contractions φ and ψ define a left and a right action of the elements of T^2_2 on the space T^1_1. We shall prove that these actions have equal identity elements $\overset{\circ}{\alpha}_l$ and $\overset{\circ}{\alpha}_d$.

For the coordinates of the left identity element $\overset{\circ}{\alpha}_l$ defined by

$$\bigwedge_{\alpha \in T^1_1} (\overset{\circ}{\alpha}_l\, a = a)$$

or, in coordinate form

$$u^{ip}_{jq}(\overset{\circ}{\alpha}_l)\, s^q_p(a) = s^i_j(a),$$

we obtain

$$u^{ik}_{jr}(\overset{\circ}{\alpha}_l) = \delta^i_r\, \delta^k_j \qquad 1 \leqslant i, j, k, r \leqslant n. \tag{3}$$

In the same way, the definition

$$\bigwedge_{a \in T^1_1} (a\ \overset{\circ}{\alpha}_d = a)$$

leads to

$$s^p_q(a)\ u^{qi}_{pj}(\overset{\circ}{\alpha}_d) = s^i_j(a). \tag{4}$$

Since

$$s^i_j(a) = \delta^i_p\, \delta^q_j\, s^p_q(a),$$

(4) leads to

$$u^{ij}_{kl}(\overset{\circ}{\alpha}_d) = \delta^i_l\, \delta^j_k \qquad 1 \leqslant i, j, k, l \leqslant n \tag{5}$$

which in view of (3) proves the identity

$$\overset{\circ}{\alpha}_l = \overset{\circ}{\alpha}_d.$$

Let h be the contraction

$$T_2^2 \times T_2^2 \to T_2^2$$

defined by

$$u_{kl}^{ij} \cdot h(\alpha, \beta) = u_{kq}^{ip}(\alpha)\, u_{pl}^{qj}(\beta) \qquad \alpha, \beta \in T_2^2$$

or, if we denote

$$h(\alpha, \beta) = \alpha\,\beta,$$

$$u_{kl}^{ij}(\alpha\,\beta) = u_{kq}^{ip}(\alpha)\, u_{pl}^{qj}(\beta).$$

It is easily seen that herewith a left and a right action on T_2^2 for each its element is defined. If we ask for the corresponding identity elements (left and right identity) of these actions, defined by

$$\bigwedge_{a \in T_2^2} (\overset{\circ}{\alpha}{}^l \alpha = \alpha) \qquad \text{and} \qquad \bigwedge_{a \in T_2^2} (\alpha\, \overset{\circ}{\alpha}{}^d = \alpha),$$

we can easily obtain

$$u_{kr}^{ij}(\underset{\circ}{\alpha}{}^l) = \delta_r^i \delta_k^j \quad \text{and} \quad u_{kr}^{ij}(\underset{\circ}{\alpha}{}^d) = \delta_r^i \delta_k^j\,,$$

and hence

$$\underset{\circ}{\alpha}{}^l = \underset{\circ}{\alpha}{}^d = \overset{\circ}{\alpha}_l = \overset{\circ}{\alpha}_d\,. \tag{7}$$

In view of (7) we will denote all the four obtained identity elements by $\overset{\circ}{\alpha}$.

It is clear that if the elements of T_1^1 are expressed by their n^2 coordinates, they can be written in a natural way as an $n \times n$-matrix. We define now a one to one mapping $\check{f}$ which maps T_2^2 onto the space $\check{R}^{n^2}$ whose elements are all $(n^2 \times n^2)$ — matrices in the following way:

To each $\alpha \in T_2^2$ whith coordinates $\alpha_{kl}^{ij} = u_{kl}^{ij}(\alpha)$ the function $\check{f}$ asigns the matrix

$$\check{\alpha} = \check{f}(\alpha) \in \check{R}^{n^2}$$

defined by

$$\check{\alpha}. = \left\| \begin{matrix}
\alpha^{11}_{11} & \alpha^{12}_{11} & \dots & \alpha^{1n}_{11} & \alpha^{11}_{12} & \alpha^{12}_{12} & \dots & \alpha^{1n}_{12} & \dots & \alpha^{11}_{1n} & \dots & \alpha^{1n}_{1n} \\
\alpha^{11}_{21} & \alpha^{12}_{21} & \dots & \alpha^{1n}_{21} & \alpha^{11}_{22} & \alpha^{12}_{22} & \dots & \alpha^{1n}_{22} & \dots & \alpha^{11}_{2n} & \dots & \alpha^{1n}_{2n} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\alpha^{11}_{n1} & \alpha^{12}_{n1} & \dots & \alpha^{1n}_{n1} & \alpha^{11}_{n2} & \alpha^{12}_{n2} & \dots & \alpha^{1n}_{n2} & \dots & \alpha^{11}_{nn} & \dots & \alpha^{1n}_{nn} \\
\alpha^{21}_{11} & \alpha^{22}_{11} & \dots & \alpha^{2n}_{11} & \alpha^{21}_{12} & \alpha^{22}_{12} & \dots & \alpha^{2n}_{12} & \dots & \alpha^{21}_{1n} & \dots & \alpha^{2n}_{1n} \\
\alpha^{21}_{21} & \alpha^{22}_{21} & \dots & \alpha^{2n}_{21} & \alpha^{21}_{22} & \alpha^{22}_{22} & \dots & \alpha^{2n}_{22} & \dots & \alpha^{21}_{2n} & \dots & \alpha^{2n}_{2n} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\alpha^{21}_{n1} & \alpha^{22}_{n1} & \dots & \alpha^{2n}_{n1} & \alpha^{21}_{n2} & \alpha^{22}_{n2} & \dots & \alpha^{2n}_{n2} & \dots & \alpha^{21}_{nn} & \dots & \alpha^{2n}_{nn} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\alpha^{n1}_{11} & \alpha^{n2}_{11} & \dots & \alpha^{nn}_{11} & \alpha^{n1}_{12} & \alpha^{n2}_{12} & \dots & \alpha^{nn}_{12} & \dots & \alpha^{n1}_{1n} & \dots & \alpha^{nn}_{1n} \\
\alpha^{n1}_{21} & \alpha^{n2}_{21} & \dots & \alpha^{nn}_{21} & \alpha^{n1}_{22} & \alpha^{n2}_{22} & \dots & \alpha^{nn}_{22} & \dots & \alpha^{n1}_{2n} & \dots & \alpha^{nn}_{2n} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\alpha^{n1}_{n1} & \alpha^{n2}_{n2} & \dots & \alpha^{nn}_{n1} & \alpha^{n1}_{n2} & \alpha^{n2}_{n2} & \dots & \alpha^{nn}_{n2} & \dots & \alpha^{n1}_{nn} & \dots & \alpha^{nn}_{nn}
\end{matrix} \right\|$$

Let $A = (\alpha^i_j)$ $(1 \leqslant i, j \leqslant n)$ be the matrix with elements $a^i_j = s^i_j(a)$, assigned to an arbitrary element $a \in T^1_1$. If we denote by $\check{k}$ the mapping which to each A assigns the $(n^2 \times 1)$ matrix

$$\check{k}(A) = \check{a} = \left\| \begin{matrix} a^1_1 \\ a^1_2 \\ \vdots \\ a^1_n \\ a^2_1 \\ a^2_2 \\ \vdots \\ a^2_n \\ \vdots \\ a^n_1 \\ a^n_2 \\ \vdots \\ a^n_n \end{matrix} \right\|,$$

it is easily seen that contraction

$$\varphi : (\alpha, a) \to \alpha\, a$$

can be interpreted as the usual matrix multiplication of $\check{\alpha}$ and $\check{a}$, namely

$$\check{f} \cdot \varphi\, (\alpha, a) = \check{\alpha}\ \check{a}. \tag{8}$$

Let $\check{G}$ be the subset of $\check{f}(T_2^2)$ whose elements are nonsingular, and let

$$G = \{\alpha;\ \check{f}(\alpha) \in \check{G}\}. \tag{9}$$

If we denote by λ_α the left action of an element $\alpha \in G$ on T_1^1,

$$\lambda_\alpha\, a = \alpha\, a \quad a \in T_1^1,$$

(8) and (9) imply for each $\alpha \in G$ the existence of some $\beta \in G$ such that

$$\lambda_\alpha \cdot \lambda_\beta = \lambda_\beta \cdot \lambda_\alpha = \lambda_\alpha^\circ,$$

whereby β is defined by the condition

$$\check{\beta} = \check{\alpha}^{-1}.$$

If we denote

$$\lambda_\beta = \lambda_\alpha^{-1}$$

and hence

$$\lambda_\alpha^{-1} = \lambda_{\alpha^{-1}},$$

we can easily prove the correctness of the relation

$$\lambda_\alpha\, \lambda_\beta = \lambda_{\alpha\beta} \quad \alpha, \beta \in T_2^2,$$

as well as the fact that for each $\alpha \in G$ the element $\alpha^{-1} \in G$ defined by

$$\check{f}(\alpha^{-1}) = \check{\alpha}^{-1} \tag{10}$$

is a uniue one. On the other hand, it is clear that the mapping

$$h : T_2^2 \times T_2^2 \to T_2^2$$

satisfies the associative law,

$$(\alpha\, \beta)\, \gamma = \alpha\, (\beta\, \gamma) \qquad \alpha, \beta, \gamma \in T_2^2.$$

These results prove that the left action λ_α of each $\alpha \in G$ on T_1^1 is a group action, i.e. G is a group whereby the group operation is the multiplication

$$(\alpha, \beta) \longrightarrow \alpha\beta$$

defined by h. The identity element is $\overset{\circ}{\alpha}$, and for each $\alpha \in G$ the inverse element α^{-1} is defined by (10). It is ease to see that G is isomorphic whith the Lie group $GL\ (n^2;\ R)$.

We shall find now the coordinate expression for the canonical form on G. If we denote by $D^1\ (\alpha)$ the tangent space of G at α, each element

$$\vec{\eta}_\alpha \in D^1(\alpha)$$

can be considered as the value at $\alpha \in G$ of some vector field $\vec{\eta}$ on T_2^2 which is left-invariant for the action of the group operation and which value $\vec{\eta}_\beta$ at each $\beta \in G$ is defined by

$$\vec{\eta}_\beta = d\lambda_\gamma \cdot \vec{\eta}_\alpha, \tag{11}$$

whereby $\beta = \gamma\,\alpha$. The canonical form μ can be defined as the mapping which to each vector $\vec{\eta}_\alpha$ at arbitrary $\alpha \in G$ assigns the value $\vec{\eta}_0$ at the identity $\overset{\circ}{\alpha}$ of the left-invariant vector field $\vec{\eta}$ induced from $\vec{\eta}_\alpha$ by (11), i.e.

$$\mu\,(\vec{\eta}_\alpha) = d\lambda_{\alpha^{-1}} \cdot \vec{\eta}_\alpha. \tag{12}$$

In order to obrain the coordinate expression for μ, we first obtain such an expression for the differential of the left action λ_α.

One of the basic consequences of the definition of vector fields and tangent vectors is that the actions of the differential forms du^{ij}_{kl} (u^{ij}_{kl} the natural coordinate system) on a field $\vec{\eta}$ gives at each α the coordinates of the vector field value at that point, namely

$$(d\,u^{ij}_{kl} \cdot \vec{\eta})_\alpha = \vec{\eta}_\alpha\,(u^{ij}_{kl}). \tag{13}$$

In our case we obtain

$$du^{ij}_{kl}\,(d\lambda_\gamma \cdot \vec{\eta}_\alpha) = \vec{\eta}_\alpha\,(u^{ij}_{kl} \circ \lambda_\gamma)$$

which in view of the identity

$$(u^{ij}_{kl} \circ \lambda_\gamma)\,\varepsilon = u^{ip}_{kq}\,(\gamma)\,u^{qj}_{pl}\,(\varepsilon) \qquad \varepsilon \in G$$

gives

$$du^{ik}_{jr}\,(d\lambda_\gamma \cdot \eta_\alpha) = u^{ip}_{jq}\,(\gamma)\, du^{qk}_{pr}\,(\vec{\eta_\alpha}). \tag{14}$$

If we denote $\partial/\partial\, u^{ik}_{jl} = \partial^{jl}_{ik}$, we have

$$d\lambda_\gamma\,(\vec{\eta_\alpha}) = du^{ik}_{jl}\,(d\lambda_\gamma \cdot \vec{\eta_\alpha})\,(\partial^{jl}_{ik})_\beta, \qquad \beta = \gamma\alpha$$

or, in view of (14),

$$d\lambda_\gamma\,(\vec{\eta_\alpha}) = u^{ip}_{jq}\,(\gamma)\, du^{qk}_{pl}(\vec{\eta_\alpha})(\partial^{jl}_{ik})_\beta. \tag{15}$$

If in (15) we fix $\gamma = \alpha^{-1}$ and hence $\beta = \overset{\circ}{\alpha}$, and if we denote

$$(\partial^{jl}_{ik})_{\overset{\circ}{\alpha}} = e^{jl}_{ik} \qquad\qquad 1 \leqslant i, j, k, l \leqslant n,$$

(15) gives the canonical form as

$$\mu = \underset{*}{u}{}^{ip}_{jq}\, du^{qk}_{pl}\, e^{jl}_{ik}$$

whereby $\underset{*}{u}{}^{ip}_{jq}$ are defined by

$$\underset{*}{u}{}^{ik}_{jl}\,(\varepsilon) = u^{ik}_{jl}\,(\varepsilon^{-1}) \qquad\qquad \varepsilon \in G.$$

REFERENCES

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