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TWO REMARKS ON RELATIVE INVERSES

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Let X, Y be Banach spaces, $L(X, Y)$ the space of all bounded linear operators from X to Y, $L(X, X)=L(X)$, $F(X, Y)$ the linear subspace of all finite *rank operators*, and let C be a complex domain. Then a holmorphic function $A: C \rightarrow F(X, Y)$ has a holomorphic *relative inverse* on C if and only if dim Im $A(\lambda), \lambda \in C$, is constant on C. This result was proven in [1]; in this note we present a different proof of that result. Furthermore, we compare the notions of *relative inverse* and of regulator as defined below. This material is contained in [3].

Recall, first, the basic defitinitions and facts.

Definition 1. If $A \in L(X, Y)$, then $B \in L(Y, X)$ is a relative in verse of A if and only if $ABA=A$, $BAB=B$.

An operator A is relatively invertible if and only Ker A and Im A are complemented subspaces of X and Y, respectively. For each pair of decompositions $X=Ker\ A \oplus X_1$, $Y=Im\ A \oplus Y_1$, there is precisely one relative inverse B with the properties $Ker\ B=Y_1$, Im $B=X_1$; and, conversely, for each relative inverse B of A, the spaces X and Y are decomposed in the described manner. If B is a relative inverse of A, then AB is the projector of Y onto Im A along Ker B, and BA is the projector of X onto Im B along Ker A.

Definition 2. Let $\Sigma(X)$ be the set of all linear (closed or not) subspaces of X. A subspace-valued function $S: C \rightarrow \Sigma(X)$ is said to be holomorphic at λ_0, $\lambda_0 \in C$, if there exists a neighborhood V of λ_0 and a projector-valued function $P: V \rightarrow L(X)$ such that (1) P is holomorphic on V, and (2) Im $P(\lambda)=S(\lambda)$, $\lambda \in V$.

If $P, Q \in L(X)$ are projectors and $\| P-O \| < 1$, then P maps Im Q isomorphically onto Im P. Using this result of B. Sz.- Nagy [6], it is easy to see (assuming V connected) that if S is holomorphic, then $S(\lambda)=S(\lambda_0)$, λ_0, $\lambda \in V$.

The following result is due to Šubin [5]; for a slightly different proof, see [3], [4].

Theorem 3. The family of subspaces $\{S(\lambda): \lambda \in G\}$ is holomorphic at λ_0 if and only if there exists a neighborhood V_1 of λ_0 and a holomorphic function $A_1: V_1 \to L(X)$ with the following properties: (1) $A_1(\lambda)$ is invertible and (2) $A_1(\lambda)\,[(S(\lambda_0)] = S(\lambda)$, $\lambda \in V_1$.

Definition 4. Let $A: G \to L(X, Y)$ be holomorphic. We say that A has a holomorphic inversi on G if there exists a holomorphic $B: G \to L(Y, X)$ such that $B(\lambda)$ is a relative inverse of $A(\lambda)$ for each $\lambda \in G$.

The following theorem provides a criterion for the existence of a holomorphic relative inverse.

Theorem 5 **([3], [4]).** Let $A: G \to L(X, Y)$ be holomorphic. Then the following statements are equivalent; (1) A has a holomorphic relative inverse on G; (2) $\lambda \to \mathrm{Ker}\ A(\lambda)$ is locally holomorphic on G and $A(\lambda)$ has a relative inverse for each $\lambda \in G$; (3) $\lambda \to \mathrm{Im}\ A(\lambda)$ is locally holomorphic on G and $A(\lambda)$ has a relative inverse for each $\lambda \in G$.

Observe that, since a subspace of finite dimension or of a finite codimension in a Banach space has a (topological) comlement, every operator $A \in F(X, Y)$ has a relative inverse.

Theorem 6. Let $A: G \to F(X, Y)$ be holomorphic. Then A has a relative holomorphic relative inverse on G if and only if dim Im $A(\lambda)$ is constant on G.

Proof. The „only if" part follows from the comment after Definition 2. For the converse, let $\lambda_0 \in G$ and let $X = \mathrm{Ker}\ A(\lambda_0) \oplus X_1$, $Y = Im\ A(\lambda_0) \oplus Y_1$; let B_0 be the (unique) relative inverse of $A(\lambda_0)$ corresponding to these decompositions. Consider the operator $A_2(\lambda) = I_Y - (A(\lambda_0) - A(\lambda))B_0$, $A_2(\lambda_0) = I_Y$ and thus is invertible in a neighborhood V_2 of λ_0. On the other hand $Im\ A(\lambda_0) = \mathrm{Ker}\,(I_Y - A(\lambda_0)\,B_0)$, so that $A_2(\lambda)(I_m A(\lambda_0)) = A(\lambda) B_0 (I_m A(\lambda_0)) \subset I_m A(\lambda)$. Since dim $A(\lambda)$ is finite and constant, the invertibility of A_2 implies $A_2(\lambda)(I_m A(\lambda_0)) = I_m A(\lambda)$ for $\lambda \in V_2$. Now A_2 is obviously holomorphic, so that, by theorem 3, the function $\lambda \to Im\ A(\lambda)$ is holomorphic at λ_0 and thus locally holomorphic on G. By theorem 5 the function A has a holomorphic inverse on G.

Turning to the notion of regulator, let us first introduce some notation. Let

$$\Phi^r_+(X, Y) = \{A \in L(X, Y): \dim \mathrm{Ker}\ A < \infty,\ I_m A \text{ complemented}\}$$

$$\Phi^r_-(X, Y) = \{A \in L(X, Y): \mathrm{Ker}\ A \text{ complemented},$$

$$\text{codim } Im\ A < \infty\}$$

$$\Phi(X, Y) = \Phi^r_+(X, Y) \cap \Phi^r_-(X, Y).$$

If A belongs to one of these classes, i.e., if A is projective semi-Fredholm of the first or the second kind, or Fredholm, then A is relatively invertible.

Definition 7 ([2], [7]). Let A, C, $D \in L(X)$. The operator C is called a *let regulator[of A* if there exists a compact operator $K_1 \in L(X)$, such that $CA - I = K_1$. The operator D is called a *right regurator of A* if there exists a compact operator $K_2 \in L(X)$, such that $AD - I = K_2$. An operator which is in the same time a left and a right regulator of A is called a *regulator of A*.

Thus, an operator possesses a (left, right) regulator if and only if its canonical image in the Calkin algebra ($\equiv L(X)/K(X)$, where $K(X)$ is the ideal of all compact operators) is (left, right) invertible.

The following result is well known; the image of $A \in L(X)$ in the Calkin algebra is denoted by $\bar{A}$, $\Phi_+^r(X) = \Phi_+^r(X, X)$, etc.

Theorem 8. The following equalities hold:

$$\{A \in L(X) : \bar{A} \text{ is left invertible}\} = \Phi_+^r(X),$$

$$\{A \in L(X) : \bar{A} \text{ is right invertible}\} = \Phi_-^r(X),$$

$$\{A \in L(X) : \bar{A} \text{ is invertible}\} = \Phi(X).$$

In other words, the classes of operators in $L(X)$ that have (left. right) regulators coincide with ($\Phi_+^r(X)$, $\Phi_-^r(X)$) $\Phi(X)$.

Therefore, any operator having a (left, right) regulator has also a relative inverse. The converse. is not true: a finite rank operator cannot have a regulator (since it is not semi-Fredholm, unless $\dim X < \infty$). We now show that relative inverses are regulators, when the latter exist.

Lema 9. Let A, B belong to $L(X)$, and let B be a relative inverse of A. Then,

(a) If $A \in \Phi_+^r(X)$, B is a left regulator of A

(b) If $A \in \Phi_-^r(X)$, B is a right regulator of A

Proof. (a) The operator $I - BA$ is a projector onto Ker A; thus $BA - I$ has finite rank.

(b) The operator $I - AB$ is a projector onto Ker B; since $B \in \Phi_+^r(X)$, $AB - I$ has finite rank.

We now show that a relative inverse is, in a certain sense, the best regulator a (semi-Fredholm) operator can have.

Lema 10. Let A, B, C belong to $L(X)$, let B be a relative inverse of A, and let C be a left regulator of A. Then,

$$Im\,(CA - I) \supset Im\,(BA - I) = \text{Ker } A.$$

Proof. Clearly, $(CA-I)$ (Ker A) $= -$Ker $A=$Ker A, so that *Im* $(CA-I) \supset$ Ker A.

Lemma 11· Let A, B, C belong to $L(X)$, let B be a relative inverse of A, and let D be a right regulator of A. Then, *Im* $A=$Ker $(AB-I) \supset$ Ker $(AD-I)$.

Proof. Let $X \varepsilon$ Ker $(AD-I)$. Then $ADx=x$. Apply the operator AB: $ABADx=ABx$, or $ADx=ABx$. Therefore $x=ABx$ and $x \varepsilon$ *Im* A.

We see that (in the notation of Lemmas 10 and 11) the operator $BA-I$ is tke „closest" to the zero operator among all the operators $CA-I$; and that the operator $AB-I$ is the closest to the zero operator among all the operators $AD-I$. The relative inverse is, in a sense, a measure of how far an operator is from being (left, right) invertible.

Let G be a domain in the complex plane and $A: G \to L(X, Y)$ a holomorphic operator-valued function. A necessary condition for A to have a holomorphic relative inverse is the constancy of dim Ker $A(\lambda)$ and codim *Im* $A(\lambda)$ on G.

Theorem 12. ([3], [4]). Let $A: G \to \Phi^r_+(X, Y)$ [respectively, $\Phi^r_-(X, Y)$] be holomorphic. Then A has a holomorphic relative inverse $B: G \to \Phi^r_-(X, Y)$ [respectively), $\Phi^r_+(X, Y)$] if and only if dim Ker $A(\lambda)=$constant [respectively, codim *Im* $A(\lambda)=$constant].

Theorem 13 ([2]; see also [7]). Let $A: G \to \Phi^r_+(X)$ [respectively, $\Phi^r_-(X)$] be holomorphic. Then there exists a holomorphic left [respectively, right] regulator

$C: G \to \Phi^r_-(X)$ [respectively, $\Phi^r_+(X)$].

The constancy of dim Ker $A(\lambda)$ [respectively, codim *Im* $A(\lambda)$] on G is not needed in Theorem 13, and it is essential in Theorem 12. However, in that case Theorem 12 is more precise: the holomorphic regulator of Theorem 13 can be built from the „best" regulators of A.

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РЕЗИМЕ

Нека се X, $У$ банахови простори, $L(X, У)$ просторот од сите ограничени линеарни оператори од X во $У$, $L(X, X) = L(X)$, $F(X, У)$ линеарниот потпростор од сите конечни рангови оператори, и нека е G комплексна област. Тогаш холоморфна функција $A : G \to F(X, У)$ има холоморфна релативна инверзна операторска функција на G ако и само ако $dim\ Im\ A(\lambda)$, $\lambda \in G$, е константна на G. Овој резултат е докажан во [1]; во оваа статија изнесуваме друг доказ на овој резултат. Понатаму, споредени се значењата на релативна инверзна операторска функција и на регулатор како што е деринирано подолу. Овој материјал се содржи во [3].