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RELATION BETWEEN A SEQUENCE OF ANALYTIC REPRESENATIONS AND THE CORESPONDING SEQUENCE OF DISTRIBUTIONS

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In this article the space of infinitely differentiable functions with compact support on the- real line is denoted by $D\ (R)$. The space of distributions is denoted by $D'\ (R)$.

It is known that for every distribution $T \in D'\ (R)$ there exists a complex function f (z) $(z=x+iy)$ analytic in the whole plane minus the support of T and such that

$$f\ (x+i\ \varepsilon)-f\ (x-i\ \varepsilon)\ \underrightarrow{weakly}\ T\ ([1],\ 5\cdot 9).$$

The function $f(z)$ is called the analytic representation for the distribution T.

Here we consider a sequence $\{f_n(z)\}$ of analytic representations for a given sequence $\{T_n\}$ of distributions. The function $f_n(z)$ is an analytic representation for T_n.

Theorem. Let $\{f_n\ (z)\}$ be a sequence of complex functions analytic for $\operatorname{Im} z \neq 0$ and which are analytic representations to the corresponding members of the sequence $\{T_n\}$ of distributions. Suppose that $f_n(z) \to f\ (z)$ $(\operatorname{Im} z \neq 0)$ and that the convergence is uniform on each compact subset. Let for a given $\delta > 0$ there exists $\varepsilon_0 > 0$ and N such that

$$|I_{n,\varepsilon}\ (\varphi) - <T_n,\ \varphi>| < \delta \qquad \text{for } 0 < \varepsilon \leqslant \varepsilon_0,$$

$$\text{where } \varphi(x) \in D\,(R),\ \ if\ \ n > N,\ \ I_{n,\varepsilon}\ (\varphi) = \int_{-\infty}^{\infty} [f_n\ (x+i\ \varepsilon) -$$

$$- f_n\ (x-i\ \varepsilon)]\ \varphi\ (x)\ dx$$

In that case

(a) The sequence $\{T_n\}$ is convergent.

(b) The function $f(z)$ is an anualytic representation for the distribution $T = \lim T_n$.

Proof. It is clear that the function $f(z)$ is analytic for $I_m z \neq 0$. We now show that the sequence $\{T_n\}$ is convergent.

$$|< T_n, \varphi > - < T_m, \varphi >| = |< T_n, \varphi > - I_{n,\varepsilon}(\varphi) - I_{m,\varepsilon})\varphi) + I_{n,\varepsilon}(\varphi) +$$

$$+ I_{m,\varepsilon}(\varphi) - < T_m, \varphi >| \leqslant | I_{n,\varepsilon}(\varphi) - < T_n, \varepsilon >| + | I_{m,\varepsilon}(\varphi) -$$

$$- < T_m, \varphi >| + |\int_{-\infty}^{\infty} [f_n(x+i\varepsilon) - f_m(x+i\varepsilon)] \varphi(x)\, dx| + |$$

$$+ | \int_{-\infty}^{\infty} [f_n(x-i\varepsilon) - f_m(x-i\varepsilon)] \varphi(x)\, dx|$$

Let $\delta > 0$, be given From conditions of the theorem it follows that for $\varepsilon \leqslant \varepsilon_0$ and $m, n \geqslant N$

$$|I_{n,\varepsilon}(\varphi) - < T_n, \varphi >| < \delta, \quad |I_{i,\varepsilon}(\varphi) - < T_m, \varphi >| < \delta, \; \varphi(x) \in D(R)$$

The uniform convergence of the sequence $\{f_n(z)\}$ implies that for fixed $\varepsilon \leqslant \varepsilon_0$ and givcn $\varphi(x)$ we can choose N such that

$$|f_n(x+i\varepsilon) - f_m(x+i\varepsilon)| < \delta, \; |f_n(x-i\varepsilon) f_m(x-i\varepsilon)| <$$

$$< \delta, \text{ if } m, n \geqslant N \tag{1}$$

Consequenly we obtain

$$| \int_{-\infty}^{\infty} [f_n(x+i\varepsilon) - f_m(x+i\varepsilon)] \varphi(x)\, dx| \leqslant \int_{-c}^{c} f_n(x+i\varepsilon)| -$$

$$- f_m(x+i\varepsilon) | \, |\varphi(x)|(x)|dx \leqslant \delta \int_{-\infty}^{\infty} |\varphi(x)|\, dx \leqslant 2c\delta M$$

where $M = \sup |\varphi(x)|$, supp $\varphi \subset [-c, c,]$, and also

$$\left| \int_{-\infty}^{\infty} [f_n(x-i\varepsilon) - f_m(x-i\varepsilon)] \varphi(x) \right| \leqslant 2c\delta M$$

Finally

(*) $|<T_n, \varphi> - <T_m, \varphi>| < 2\delta + 4c\delta M$, *for* $m, n \geqslant N$, $\varphi(T) \in D(R)$

From (*) it follows that $\{T_n\}$ is a Cauchy sequence of distributions and consequently *there* exists a distribution $T = \lim T_n$.

(*b*). Next we show that the function f (z) is an analytic representation for the distribution T.

Let a $\delta > 0$ be given.

$$\left| <T, \varphi> - \int_{-\infty}^{\infty} [f(x+i\varepsilon) - f(x-i\varepsilon)] \; \varphi(x)\, dx \right| =$$

$$= |<T, \varphi> - <T_n, \varphi> + <T_n, \varphi> - I_\varepsilon(\varphi) - I_{n,\varepsilon}(\varphi) + I_{n,\varepsilon}(\varphi)|$$

Where $\varepsilon \leqslant \varepsilon_0$ · *Let N be such that*

$|<T_n, \varphi> - I_{n,\varepsilon}(\varphi)| < \delta$, $|<T, \varphi> - <T_n, \varphi>| < \delta$ $(T = \lim T_n)$

$|I_{n,\varepsilon}(\varphi) - I_\varepsilon(\varphi)| < \delta$, (*for a* given $\varphi(x) \in D(R)$ and fixed $\varepsilon \leqslant \varepsilon_0$)

$f_n(x+i\varepsilon) - f_n(x-i\varepsilon)$ uniformly on sup φ to $f(x+i\varepsilon) - f(x-i\varepsilon)$,

Hence we have

$$\left| <T, \varphi> - \int_{-\infty}^{\infty} [f(x+i\varepsilon) - f(x) - i\varepsilon)] \, \varphi(x)\, dx \right| < 3\,\delta.$$

as ε is arbitrary from $[0, \varepsilon_0]$ we conclude

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} [f(x+i\varepsilon) - f(x-i\varepsilon)]\, \varphi(x)\, dx = <T, \varphi>.$$

This completes the proof.

As an illustration we consider the regular distribution $T = [e^{-t^2}]$ The analytic representation for the distribution $[e^{-t^2}]$ is the function

$$\hat{T}(z) = \frac{1}{2\pi i} < T_{t}, \frac{1}{t-z} > = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}\, dt, \ \lim \neq 0.$$

The last integral may be computed by the Fourier transform technique ([1], 8.25), Here we consider the polynomials

$$P_n\ (t)=1-\frac{t^2}{1!}+\frac{t^4}{2!}+\ldots+(-1)\,\frac{t^{2n}}{n!}$$

Let $T_n=[P_n(t)]$. The analytic representation for the distribution T_n is the function

$$f_n(z)=\begin{cases}\ \ \dfrac{1}{2}\,P_n(z), & Im\, z>0\\ -\dfrac{1}{2}\,P_n\ (z) & Im\, z<0\end{cases}$$

It is easy to prove that the sequence of functions $\{f_n\ (z)\}$ and the sequence of distributions $\{T_n\}$ satisfy the conditions of the theorem. Since

$$\lim f_n\,(z)=\begin{cases}\ \ \dfrac{1}{2}\,e^{-z^2}, & Im,\ z>0\\ -\dfrac{1}{2}\,e^{-z^2}, & Im\ \ z<0\end{cases}$$

it follows that the analytic representation for the distribution $[e^{-t^2}]$ is the function

$$f\ (z)=\begin{cases}\ \ \dfrac{1}{2}\,e^{-z^2}, & Im\, z>0\\ -\dfrac{1}{2}\,e^{-z^2}, & Im\ \ z<0\end{cases}$$

It is clear that in the same way the analytic representation may be computed for $[e^{-t^4}]$ e tc.

REFERENCES

[1]. Bremermann H., Распределенија, комплексние премение и преобразованија Furie, издателство „Мир", Москва 1968.

[2] Garmichael R. D-, Representation of distributiones in O_α as boundary values of functions in tube domains, Гласник математички том (27), 1972.

[3] Владимиров В, С. Уравненија математическој физики, „Наука" Москва 1872.

ЕДНА РЕЛАЦИЈА МЕЃУ НИЗА ОД АНАЛИТИЧКИ РЕПРЕЗЕНТАЦИИ И СООДВЕТНА НИЗА ОД ДИСТРИБУЦИИ

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Резиме

Во оваа работа е покажано следното:

Ако $\{f_n(z)\}$ е низа од аналитични репрезентации за соодветните елементи од низата дистрибуции $\{T_n\}$ и ако $f(z) = \lim\limits_{n\to\infty} f_n(z)$, при што конвергенцијата е рамномерна на компактни множества; тогаш постои дистрибуција $T=\lim\limits_{n\to\infty} T_n$ чија што реперезентација е функцијата $f(z)$. Резултатот е илустриран со еден пример.