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ON A DUAL PAIR OF LP-PROBLEMS

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The solution of the linear complementarity problem obtained in [1[*is used for solving a dual pair of LP-problems.*

We use the following notational conventions. Let N denote the set of integers $\{1, 2, \ldots, n \geqslant 2\}$. If $I \subseteq N$, then $\underline{I} = N - I$. The identity matrix of order k is denoted by $E^{(k)}$. A matrix whose elements are all one is denoted by E. Given an nxn-matrix A and subsets I, J of N, let A_{IJ} denote the submatrix obtained form A by deleting all the rows corresponding to $\underline{I}$ and all the columns corresponding to $\underline{J}$. Instead of A_{IJ} we write simply

$$A_I \text{ if } I = J,$$

$$(A)_i \text{ if } I = \{i\},\ J = N,$$

$$(A)^j \text{ if } I = N,\ J = \{j\},$$

$$(A)_{ij} \text{ if } I = \{i\},\ J = \{j\}.$$

Similarly, if x is an n-vector, then x_I denotes the subvector whose components are x_i, $i \in I$. We denote by M the nxn matrix whose elements are

$$(M)_{ij} = \begin{cases} \quad a, & j \in N,\ i = \pi(j) \\ -1, & j \in N,\ i \neq \pi(j) \end{cases}$$

where π is a given permutation on N, and a is a given real greater than $n-2$

Given a subset I of N with k $(0 \leqslant k \leqslant n)$ indices, let B be the matrix

$$B = \begin{bmatrix} M_I & 0 \\ M_{\underline{I}I} & E^{(n-k)} \end{bmatrix}$$

There are two cases when B is nonsingular.

Case I. If $\pi(j) \in I$, $j \in I$, then M_I is of the same type as M. So, if $k \neq a+1$, then M_I is a nonsingular matrix,

$$(a+1)(a+1-k)(M_I^{-1})_{ij} = \begin{cases} a+2-k, & i \in I,\ j = \pi(i) \\ 1, & i \in I,\ j \in I - \{\pi(i)\} \end{cases}, \quad i, j \in I,$$

and it can easily be found that

$$B^{-1} = \begin{bmatrix} M_I^{-1} & 0 \\ -\dfrac{1}{a+1-k} E & E^{(n-k)} \end{bmatrix} \tag{1}$$

Case II. If there exists an index $l \in I$ such that

$$\pi(l) \notin I \text{ and } \pi(j) \in I,\ j \in I - \{l\}$$

then there exists an index $m \in I$ such that

$$(M_I)_m = -[1 \ldots 1].$$

Also M_I is nonsingular, and for $i, j \in I$

$$(a+1)(M_I^{-1})_{ij} = \begin{cases} -(a+2-k), & i = l,\ j = m \\ -1, & i = l,\ j \neq m \\ -1, & i \in I - \{l\},\ j = m \\ 1, & i \in I - \{l\},\ j = \pi(i) \\ 0, & \text{in any other case.} \end{cases}$$

Again we can easily find

$$B^{-1} = \begin{bmatrix} M_I^{-1} & 0 \\ -M_{\underline{I}I} M_I^{-1} & E^{(n-k)} \end{bmatrix},$$

where

$$(M_{\underline{I}I} M_I^{-1})_{ij} = \begin{cases} -(a+1-k), & i = \pi(l),\ j = m \\ -1, & i = \pi(l),\ j \in I - \{m\} \\ 1, & i \in \underline{I} - \{\pi(l)\},\ j = m \\ 0, & \text{in any other case} \end{cases}$$

Now we consider the dual pair of LP-problems

$$(P) \qquad \min\ \{\ pz \mid q + Mz \geqq 0,\ z \geqq 0\}$$

$$(D) \qquad \max\ \{-vq \mid p - vM \geqq 0,\ v \geqq 0\}$$

where p, q are given n-vectors, and z, v are the n-vectors of variables. (We notice that the dual pair (P), (D) is equivalent to the more general dual pair

$$(P') \qquad \min \{p' z' \mid q' + DMF\, z' \geqq 0,\ z' \geqq 0\}$$

$$(D') \qquad \max \{- v' q' \mid p' - v' DMF \geqq 0,\ v' \geqq 0\}$$

where D and F are given positive diagonal matrices. Indeed, by the substitutions $z = F^{-1} z'$, $v = v' D^{-1}$, $q = Dq'$, $p = p' F$ the pair (P'), (D') becomes (P), (D)).

It is obvious that if $q \geqq 0$ and $p \geqq 0$, then $z = 0$, $v = 0$ is a pair of optimum solutions to (P) and (D), respectively.

Suppose that $q \not\geq 0$ and $p \not\geq 0$ (if $q \geqq 0$ and $p \not\geq 0$, or $p \geqq 0$ and $q \not\geq 0$, then we immediately have a feasible initial basis to (P), or (D), and we can make the conclusions as below discussing either (D), or (P) only).

Let the indices r_i, $l_i \in N$, $i = 1, 2$ be defined as follows:

$$q_{r_1} = \max_{j \in N} \{q_j\}, \qquad p_{r_2} = \min_{j \in N} \{p_j\},$$

$$l_1 = \pi^{-1}(r_1), \qquad l_2 = \pi(r_2),$$

and denote

$$s_1 = \sum_{j \in N} q_j + (a + 1 - n)\, q_{r_1}, \quad s_2 = \sum_{j \in N} p_j + (a + 1 - n)\, p_{r_2}$$

Introducing the n-vectors w, u of nonnegative slack vaariables

$$w = q + Mz \geqq 0, \quad u = p - v M \geqq 0$$

in (P) and (D), respectively, and applying the theorems 1, 2 [1] for the linear complementarity problems

$$(1) \qquad \begin{aligned} w &= q + Mz \\ w, z &\geqq 0 \\ w^T z &= 0, \end{aligned}$$

$$(2) \qquad \begin{aligned} u^T &= p^T - M^T v^T \\ u, v &\geqq 0 \\ u\, v^T &= 0 \end{aligned}$$

we obtain the following statements.

(i) If $s_i \geqslant 0$, $i = 1, 2$, then for any $a > n - 2$ both (P) and (D) have optimum solutions $\hat{z}$, $\hat{v}$, respectively.

a) For $n - 2 < a < n - 1$ $\hat{z}$ can be found by the revised simplex method starting with the basis $-M$ of (1). Alternatively, we can find $\hat{v}$ by the revised simplex method starting with the basis

$$(3) \qquad \begin{bmatrix} M_{I_2}^T & 0 \\ M_{\underline{I_2} I_2}^T & E^{(n-k)} \end{bmatrix}$$

of (2), where I_2 ($k = \text{card}\,(I_2)$) is determined applying the following algorithm:

Step 0. Initialize $\nu = 0$, $I^{(\nu)} = N - \{l_2\}$, $l^{(\nu)} = l_2$, $s^{(\nu)} = s_2$ and test $r_2 = l^{(\nu)}$.

0.1 If yes, then $I_2 = I^{(\nu)}$, $k = n - 1$, and M_{I_2} in (3) is of the same type as M_I in the case **I**; stop!

0.2 If no, go to step 1.

Step 1. Set $\nu = \nu + 1$, find $s^{(\nu)} = s^{(\nu-1)} - p_{l^{(\nu)}} + p_{r_2}$ and test $s^{(\nu)} \leqslant 0$.

1.1 If yes, then $I_2 = I^{(\nu-1)}$, $k = n - \nu$, and M_{I_2} in (3) is of the same type as M_I in the case II; stop!

1.2 If no, set $I^{(\nu)} = I^{(\nu-1)} - \{\pi\,(l^{(\nu-1)})\}$ and go to step 2.

Step 2. Test $r_2 = \pi\,(l^{(\nu-1)})$.

2.1 If yes, then $I_2 = I^{(\nu)}$, $k = n - \nu - 1$, and M_{I_2} in (3) is of the same type as M_I in the case **I**; stop!

2.2 If no, set $l^{(\nu)} = \pi\,(l^{(\nu-1)})$ and go to step 1.

b) For $a > n - 1$ $\hat{v}$ can be found by the revised simplex method starting with the basis M^T of (2). Alternatively, we can find $\hat{z}$ by the revised simplex method starting with the basis

$$(4) \qquad \begin{bmatrix} -M_{I_1} & 0 \\ -M_{I_1 I_1} & E^{(n-k)} \end{bmatrix}$$

of (1), where I_1 ($k = \text{card}\,(I_1)$) is determined applying the following algorithm:

Step 0. Initialize $\nu = 0$, $I^{(\nu)} = N$, $r^{(\nu)} = r_1$, $l^{(\nu)} = l_1$, $s^{(\nu)} = s_1$ and go to step. 1.

Step 1. Set $I^{(\nu+1)} = I^{(\nu)} - \{l^{(\nu)}\}$ and test $r^{(\nu)} \neq l^{(\nu)}$.

1.1 If yes, then $I_1 = I^{(\nu+1)}$, $k = n - \nu - 1$ and M_{I_1} in (4) is of the same type as M_I in the case II; stop!

1.2 If no, go to step 2.

Step 2. Set $\nu = \nu + 1$, find $q_{r^{(\nu)}} = \max_{j \in I^{(\nu)}} \{q_j\}$, $l^{(\nu)} = \pi^{-1}(r^{(\nu)})$,

$$s^{(\nu)} = \sum_{j \in I^{(\nu)}} q_j + (a + \nu + 1 - n)\, q_{r^{(\nu)}} \text{ and test } s^{(\nu)} \leqslant 0.$$

2.1 If yes, then $I_1 = I^{(\nu)}$, $k = n - \nu - 1$, and M_{I_1} in (4) is of the same type as M_I in the case **I**; stop;

2.2 If no, go to step 1.

c) For $a = n - 1$ we can find $\hat{z}$ by the revised simplex method starting with a basis obtained as in (i), b). Alternatively, we can find $\hat{v}$ by the revised simplex method starting with a basis obtained as in (*i*), a).

(ii) If $s_i < 0$, $i=1, 2$, then

a) For $n - 2 < a < n - 1$ (*P*) has no feasible solution, (*D*) has a feasible solution, but its objective function is unbounded in the direction of maximization; $v = pM^{-1} + \lambda\,(-M^{-1})_{l_1}$, $\lambda \geqslant 0$, is an infinite feasible edge along which the objective function strictly increases.

b) For $a = n - 1$ both (*P*) and (*D*) fail to have feasible solutions.

c) For $a > n - 1$ (*D*) has no feaible solution, (*P*) has a feasible solution, but its objective function is unbounded in the direction of minimization; $z = -M^{-1} q + \lambda\,(M^{-1})^{l_2}$, $\lambda \geqslant 0$, is an infinite edge along which the objective function strictly decreases.

(iii) If $s_1 < 0$ and $s_2 \geqslant 0$, then

a) For $n - 2 < a \leqslant n - 1$ (*D*) has a feasible solution, but its objective function is unbounded in the direction of maximization; the end point $\bar{v} = [\bar{v}_{I_2}, o_{\bar{I}_2}]$ of an infinite feasible edge

$$v = \bar{v} + \lambda t,\ \lambda \geqslant 0, \text{ where } \quad t = \begin{cases} -(M^{-1})_{l_1} & \text{if } n-2 < a < n-1, \\ [1 \ldots 1] & \text{if } a = n-1 \end{cases}$$

along which the objective function strictly increases can be found pivoting on (3) in (2), where I_2 is determined as in (*i*) a).

b) For $a > n - 1$ $\hat{z} = -M^{-1} q$ is an optimum solution of (*P*), and $\hat{v} = pM^{-1}$ is an optimum solution of (*D*).

(iv) If $s_1 \geqslant 0$ and $s_2 < 0$, then

a) For $n - 2 < a < n - 1$ again $\hat{z} = -M^{-1} q$ is an optimum solution of (*P*), and $\hat{v} = p\, M^{-1}$ is an optimum solution of (*D*).

b) For $a \geqslant n-1$ (P) has a feasible solution, but its objective function is unbounded in the direction of minimization; the end point

$$\bar{z} = \begin{bmatrix} \bar{z}_{I_1} \\ o_{I_1} \end{bmatrix}$$

of an infinite feasible edge $z = \bar{z} + \lambda t$, $\lambda \geqslant 0$, where $t = \begin{cases} (M^{-1})^{l_2}, & a > n-1 \\ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, & a = n-1 \end{cases}$

along which the objective function strictly decreases can be found pivoting on (4) in (1), where I_1 is determined as in (i) b).

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[2] O. L. Mangasarian: Linear complementarity problems solvable by a single linear program, Mathematical Programming, V 10 (1976) No 2.

ЗА ПАР ЗАЕМНО ДУАЛНИ ЛП-ЗАДАЧИ

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Р е з и м е

Решението на линеарниот проблем на комплементарност добиено во [1] се користи за решавање пар заемно дуални ЛП-задачи.