INDUCED ACTIONS OF $GL(n;R)$ AND $L(M)$

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Let M be a differentiable manifold, dim $M=n$. The general linear group $GL(n;R)$ will be denoted by G, and the bundle of linear frames $T(L(M))$ of M by P. For the induced action of the tangent group $T(G)$ on the tangent bundle $T(P)$ we have the following results:

Let $\{x^1, \ldots, x^n\}$ be a local coordinate system in a coordinate neighborhood $U \subset M$, and $\{x^i, X_j^k\}$, $1 \leqslant i, j, k \leqslant n$, the induced coordinate system on $\pi^{-1}(U)$, where π is the natural projection $D \to M$. Let $\{s_j^i\}$, $1 \leqslant i, j \leqslant n$, be the natural coordinate system on G, where the upper index means the row, and the lower one the column of the corresponding matrix. For arbitrary $(u, a) \in \pi^{-1}(U) \times G$ and arbitrary tangent vector X on P at u we have

$$dx^i(dR_a \cdot X) = dx^i(X) \qquad 1 \leqslant i \leqslant n; \tag{1}$$

$$dX_j^k(dR_a \cdot X) = s_j^m(a)\, dX_m^k(X) \qquad 1 \leqslant k, j \leqslant n; \tag{1'}$$

$$dx^i(dL_a \cdot X) = dx^i(X) \qquad 1 \leqslant i \leqslant n; \tag{2}$$

$$dX_j^k(dL_a \cdot X) = s_m^k(a)\, dX_j^m(X) \qquad 1 \leqslant k, j \leqslant n, \tag{2'}$$

where by R_a is denoted the right action of the element $a \in G$ on P, and by L_a the corresponding left action. Then, for each tangent vector A on G at a, we obtain

$$dx^i(d\sigma_u \cdot A) = 0 \qquad 1 \leqslant i \leqslant n, \tag{3}$$

$$dX_j^k(d\sigma_u \cdot X) = X_m^k(u)\, ds_j^m(A) \qquad 1 \leqslant k, j \leqslant n, \tag{3'}$$

where for each $u \in P$ the mapping σ_u is defined by $a \in G \to ua \in P$.

Then, as a corollary from $(1), \ldots, (2')$ we have

$$dx^i(d(ad(a)) \cdot X) = dx^i(X) \qquad 1 \leqslant i \leqslant n, \tag{4}$$

$$dX_j^k(d(ad(a)) \cdot X) = s_m^k(a)\, s_j^r(a^{-1})\, dX_r^m(X) \qquad 1 \leqslant k, j \leqslant n, \tag{4'}$$

where by $ad(a)$ is denoted the mapping $u \in P \to aua^{-1}$, and from (1), ..., (3′)

$$dx^i(dR_a(d\sigma_u \cdot A)) = 0 \qquad 1 \leqslant i \leqslant n; \qquad (5)$$

$$dX_j^k(dR_a(d\sigma_u \cdot A)) = s_j^r(a)\, X_m^k(u)\, ds_r^m(A) \qquad 1 \leqslant k,\ j \leqslant n; \qquad (5')$$

$$dx^i(dL_a(d\sigma_u \cdot A)) = 0 \qquad 1 \leqslant i \leqslant n; \qquad (6)$$

$$dX_j^k(dL_a(d\sigma_u \cdot A)) = s_m^k(a)\, X_r^m(u)\, ds_j^r(A) \qquad 1 \leqslant k,\ j \leqslant n. \qquad (6')$$

From now on the tangent space of a manifold M at a point $p \in M$ will be denoted by $D^1(p)$.

Let Γ be a linear connection in P and ω the connection form of Γ. The vertical and horizontal subspaces at a point $u \in P$ will be denoted by G_u and Q_u respectively. In the foregoing notation, let $\nu\colon U \to P$ be the natural cross section of P over U defined by $x \in U \to \{(\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x\}$, i. e. $x^i(\nu(x)) = x^i(x)$ and $X_j^k(\nu(x)) = \delta_j^k$, $1 \leqslant i, j, k \leqslant n$. Let Γ_{ij}^k, $1 \leqslant i$, l, $k \leqslant n$, be the components of the linear connection Γ with respect to the jocal coorinate system $\{x^1, \ldots, x^n\}$. Theese functions are defined on U by

$$\nu^*\omega = \Gamma_{ij}^k\, dx^i\, E_k^j,$$

where E_k^j, $1 \leqslant j$, $k \leqslant n$, is the natural basis of the Lie algebra g of G. If $\Gamma_{i'j'}^{k'}$, $1 \leqslant i', j', k' \leqslant n$, are the components of Γ with respecs to a local coordinate system $\{x^{1'}, \ldots, x^{n'}\}$ in a coordinate neighborhood U' ($U \cap U' = N \neq \emptyset$), in the intersection N we have the transformation law[1])

$$\Gamma_{i'j'}^{k'} = \partial_{i'}^i \partial_{j'}^j \cdot \partial_k^{k'}\, \Gamma_{ij}^k + \partial_{i'j'}^k \partial_k^{k'} \qquad 1 \leqslant i', j', k' \leqslant n, \qquad (7)$$

where $\partial_{i'}^i = \partial x^i/\partial x^{i'}$, $\partial_i^{i'} = \partial x^{i'}/\partial x^i$ and $\partial_{i'j'}^k = \partial^2 x^k/(\partial x^{i'}\partial x^{j'})$. Using the foregoing results and (7), we prove the following wellknown theorem:

Theorem. Assume that, for each local coordinate system $\{x^1, \ldots, x^n\}$, there is given a set of functions Γ_{ij}^k, $1 \leqslant i, j, k \leqslant n$, which satisfy the transformation rule (7). Then there is a unique linear connection Γ whose components with respect to $\{x^1, \ldots, x^n\}$ are the functions Γ_{ij}^k, and the connection form $\omega = \omega_j^i E_i^j$ of Γ is given in terms of the induced local coordinate system $\{x^i, X_j^k\}$, $1 \leqslant i, j, k \leqslant n$, by

$$\omega_j^i = Y_k^i(dX_j^k + \Gamma_{rs}^k X_j^s\, dx^r) \qquad i, j = 1, \ldots, n, \qquad (8)$$

where $(Y_j^i) = (X_j^i)^{-1}$.

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1) For the proof of (7) see [2], p. 141—142.

Proof of (1), (1′), (2) and (2′). In the foregoing notation we have

$$(x^i \circ R_a)\, u = x^i (ua) = x^i (u),$$

$$(X^j_k \circ R_a)\, u = X^j_k (ua) = X^j_m (u)\, s^m_k (a)$$

and therefore

$$(dR_a \cdot X)\, x^i = X (x^i \circ R_a) = X x^i$$

proving (1), and

$$(dR_a \cdot X)\, X^i_j = X (X^i_j \circ R_a) = s^m_j (a)\, X X^i_m$$

proving (1′).

(2) and (2′) can be proved in the same way.

Proof of (3) *and* (3′). This proof is similar tn the foregoing one. From

$$(x^i \circ \sigma_u)\, a = x^i (\sigma_u a) = x^i (ua) = x^i (u),$$

$$(X^k_j \circ \sigma_u)\, a = X^k_j (\sigma_u\, a) = X^k_j (ua) = X^k_m (u)\, s^k_j (a)$$

we obtain

$$(d\sigma_u \cdot A)\, x^i = A (x^i \circ \sigma_u) = 0$$

proving (3), and

$$(d\sigma_u \cdot A)\, X^k_j = A\, (X^k_j \circ \sigma_u) = X^k_m (u)\, ds^m_j (A)$$

proving (3′).

It is ease now to obtain as a corollary the expressions (4), (4′), (5) (5′), (6) and (6′).

Proof of the theorem for linear connections. We shall use the following lemmas:

Lemma 1. The connection form ω of a connection satsfies the following conditions:

(a) $\omega\, (d\sigma^u \cdot A_e) = A$ for each $A \in g$ and $u \in P$, where by e is denoted the identity of G;

(b) $(R_a)^* \omega = \text{ad}\ (a^{-1})\, \omega$ for each $a \in G$.

Conversely, given a g-valued l-form ω on P satisfying (a) and (b), there is a unique connection Γ in P whose connection form is ω, For fhe proof see [2], p. 64, Proposition 1. 1, where this result is obtained for general connections.

Lemma 2. If $a, b \in G$ and J_q^r the vector field $\partial/\partial s_r^q$, $1 \leqslant r, q \leqslant n$, on G, we have

$$(ad(a))(J_q^r)_b = s_q^m(a)\, s_p^r(a^{-1})(J_m^p)_{(ad(a))}.$$

This result is proved in [4].

Lemma 3. Let M_i, $1 = 1, 2, 3$, be a differentiable manifold and f a mapping of the product manifold $M_1 \times M_2$ into M_3. Let $P_i \in M_i, Z_i \in D^1(P_i)$, $i = 1, 2$. If Z is the tangent vector on $M_1 \times M_2$ at (p_1, p_2) defined by $(X_1, X_2) \in D^1(p_1, p_2)$ we have

$$df \cdot Z = df_1 \cdot Z_1 + df_2 \cdot Z_2,$$

where

$$f_1 : M_1 \to M_3 \quad \text{and} \quad f_2 : M_2 \to M_3$$

are the mappings defined as

$$f_1(p) = f(p, p_2) \quad \text{for} \quad p \in M_1,$$

$$f_2(q) = f(p_1, q) \quad \text{for} \quad q \in M_2.$$

For the proof see [2], p. 11—12.

To prove the theorem, first we prove that the form ω, given by (8), satisfies the conditions (a) and of Lemma 1.

We shall hold the foregoing notations and denote the vector field $\partial/\partial X_j^i$, $1 \leqslant i, j \leqslant n$, on P by $\overline{\overline{X}}{}_i^j$, $i, j = 1, \ldots, n$. To prove (a), it is sufficient to prove the relation

$$d\sigma_u \cdot \omega(\overline{\overline{X}}{}_j^i)_u = (\overline{\overline{X}}{}_j^i)_u, \qquad u \in \pi^{-1}(U), \qquad 1 \leqslant i, j \leqslant n.$$

From (8) we obtain

$$\omega_j^i(\overline{\overline{X}}{}_s^r)_u = Y_m^i(\partial X_j^m / \partial X_r^s)_u = \delta_j^r Y_s^i(u),$$

so

$$\omega(\overline{\overline{X}}{}_s^r)_u = Y_s^m(u) E_m^r. \tag{9}$$

Then from (3) and (3′)

$$d\sigma_u \cdot E_j^i = X_j^m(u)(\overline{\overline{X}}{}_m^i)_u$$

and therefore

$$d\sigma_u \cdot \omega\,(\overline{\overline{X}}{}^i_j)_u = Y^m_j(u)\, d\sigma_u \cdot E^i_m = Y^m_j(u)\, X^p_m(u)\,(\overline{\overline{X}}{}^i_p)_u = \delta^m_j\,(\overline{\overline{X}}{}^i_m)_u = (\overline{\overline{X}}{}^i_j)_u$$

proving (a).

To prove that

$$\omega\,(dR_a \cdot X) = (ad\,(a^{-1}))\,\omega\,(X) \qquad X \in D^1(u), \tag{10}$$

it is sufficient to verify (10) in the two special cases: $X = (\partial/\partial x^i)_u$, and $X = (\overline{\overline{X}}{}^k_j)_u$, $i, j, k = 1, \ldots, n$. If we denote $X_i = \partial/\partial x^i$, $1 \leqslant i \leqslant n$, for arbitrary $a \in G$ (1) and (1′) give

$$dR_a\,(X_i)_u = (X_i)_{ua}, \quad dR_a\,(\overline{\overline{X}}{}^k_j)_u = s^{\,k}_{\,m}(a)\,(\overline{\overline{X}}{}^m_j)_{ua}. \tag{11}$$

Then from (8) we obtain

$$\omega^i_j\,(X_k) = X^m_j\, Y^i_r\, \Gamma^r_{km},$$

so

$$\omega\,(X_s)_u = \Gamma^k_{sr}(\pi\,(u))\, X^r_j(u)\, Y^i_k(u)\, E^j_i. \tag{12}$$

Using (12) and Lemma 2, we obtain

$$(ad\,(a^{-1}))\,\omega\,(X_s)_u = \Gamma^k_{sr}(\pi(u))\, X^r_q(ua)\, Y^p_k(ua)\, E^q_p$$

what with respect to (11) and (12) proves (b) for the case $X = (X_s)_u$. Then as a consequence from (9) and Lemma 2 we obtain

$$(ad\,(a^{-1}))\,\omega\,(\overline{\overline{X}}{}^i_j)_u = s^i_r(a)\; Y^m_j(ua)\; E^r_m, \tag{13}$$

and from (9) and (11)

$$\omega\,(dR_a(\overline{\overline{X}}{}^i_j)_u) = s^i_m(a)\,\omega\,(\overline{\overline{X}}{}^m_j)_{ua} = s^i_m(a)\, Y^r_j(ua)\, E^m_r. \tag{14}$$

(13) and (14) prove (b) for the case of vertical vectors, so by Lemma 1 ω defines a connection in P.

The proof that Γ^k_{ij}, $1 \leqslant k,\ i,\ j \leqslant n$, are the components of the connection defined by ω is taken from [2], p. 143:

If we consider the natural cross section ν over U, have

$$\nu^*\omega^i_j = \delta^i_k\,(d\,\delta^k_j + \Gamma^k_{mr}\,\delta^r_j\, dx^m) = \Gamma^i_{mj}\, dx^m$$

proving that Γ^k_{ij}, $1 \leqslant k,\ i,\ j, \leqslant n$, are the components of the conection defined by ω.

To prove the invariance of ω, consider the mapping $\gamma : N \to G$ defined by

$$\nu'(x) = \nu(x)\,\gamma(x) \qquad x \in N,$$

where by ν' is denoted the natural cross section over U' with respect to the local coordinate system $\{x^{1'}, \ldots, x^{n'}\}$. It is ease to see that $\gamma = (\partial^{i}_{i'})$. If we consider the mappings

$$f : (u,\ a) \in \nu(N) \times \gamma(N) \to ua,$$

$$h_{\gamma,\nu} : x \in N \to (\nu(x),\ \gamma(x))$$

and

$$h^*_{\nu,\gamma} f : x \in N \to f(\nu(x),\ \gamma(x)),$$

the last of which coincides with ν', for arbitrary $x \in N$ and $X \in D^1(x)$ we have

$$d\nu'(X) = dh^*_{\nu,\gamma} f(X) = df(d\nu \cdot X,\ d\gamma \cdot X)$$

which by Lemma 3 can be written as

$$d\nu'(X) = dR_{\gamma(x)} \cdot d\nu(X) + d\sigma_{\nu(x)} \cdot d\gamma(X) \tag{15}$$

In veiw of Lemma 1 and (15) we obtain

$$(\nu')^* \omega(X) = (ad(\gamma(x))^{-1})\, \nu^* \omega(X) + \omega(d\sigma_{\nu(x)} \cdot d\gamma(X)). \tag{16}$$

If μ is the canonical 1-form on G

$$\mu = t^i_j\, ds^j_k\, E^k_i, \text{ 1)}$$

where $(t^i_j) = (s^i_j)^{-1}$, we have

$$\gamma^*\mu = \partial^{i'}_j\, d\partial^j_{k'}\, E^{k'}_{i'} = \partial^{i'}_j\, \partial^j_{k'm'}\, dx^{m'}\, E^{k'}_{i'}\ (E^{k'}_{i'} = E^k_i), \tag{17}$$

so for the second summand of the right side of (16) we obtain

$$\omega(d\sigma_{\nu(x)} \cdot d\gamma(X)) = \omega(d\sigma_{\nu(x)\gamma(x)} \cdot \mu(d\gamma \cdot X)) = \mu(d\gamma \cdot X) = \gamma^*\mu(X).$$

Therefoıe (16) can be written as

$$(\nu')^*\omega = (ad(\partial^{i'}_i))\, \nu^*\omega + \gamma^*\mu,$$

or, by the definition of the components of a linear connection and (17)

$$(\nu')^*\omega = \Gamma^i_{jk}\, dx^j\, (ad(\partial^{i'}_i))\, E^k_i + \partial^{i'}_j\, \partial^j_{k'm'}\, dx^{m'}\, E^{k'}_{i'}. \tag{18}$$

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1) This expression is obtained for instance in [2], p. 142, and 4.

Using Lemma 2 (18) gives

$$(\nu')^*\omega = (\partial_i^{i'}\partial_{j'}^{j}\partial_{k'}^{k}\Gamma_{jk}^{i} + \partial_i^{i'}\partial_{j'k'}^{i})\, dx^{j'} E_{i'}^{k'}$$

which with respect to the transformation law of the components Γ_{ij}^k, i, j' $k = 1, \ldots, n$, can be written as

$$(\nu')^*\omega = \Gamma_{j'k'}^{i'}\, dx^{j'}\, E_{i'}^{k'}. \tag{19}$$

(19) proves that in the local coordinate system $\{x^{1'},\ldots,x^{n'}\}$ the functions $\Gamma_{i'j'}^{h'}$, i', j', $k' = 1, \ldots, n$, are the components of the linear connection defined by ω, so with respect to the uniqueness of that connection (Lemma 1), we have

$$\omega_{j'}^{i'} = Y_{k'}^{i'}(dX_{j'}^{k'} + \Gamma_{r's'}^{k'} X_{j'}^{s'}\, dx^{r'}) \qquad 1 \leqslant i',\ j' = n,$$

proving the invariance of ω.

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