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A NOTE ON NON-EXISTENCE FOR SOME CLASSES OF CONTINUOUS (3,2) GROUPS

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Abstract. In this paper it is proved that if M is n-cube ($n \geq 1$), or M is n-dimensional sphere, or M is a connected subset of R which contains more than one point, then there does not exist a continuous function $[\,]: M\times M\times M \to M\times M$ which defines a (3,2) group on M.

The definition of (n,m) groups ($n > m$) is given in [1]. We give here only the definition of (3,2) groups.

DEFINITION 1. The pair $(M, [\,])$ is (3,2) group if $[\,]: M^3 \to M^2$ and the next conditions are satisfied

(i) $(\forall_{a,b,c,d\in M})$ $\quad [[abc]d] = [a[bcd]]$

(ii) For arbitrary $a,b,c \in M$ the equations $[axy]=(b,c)$ and $[xya]=(b,c)$ have solutions for x and y.

It can be proved that the equations in (ii) have unique solutions if $(M,[\,])$ is (3,2) group.

Dimovski [3] has shown the existence of non-trivial (3,2) group by constructing the free (3,2) group. In this paper we shall give some results of non-existence for some classes of continuous (3,2) groups.

The mapping $\Psi: M^2\times M^2 \to M^2$ defined by $\Psi((a,b),(c,d)) = [[abc]d]$ induces a group structure (M^2,Ψ). If (e_1,e_2) is the identity in (M^2,Ψ) then it is proved in [2] that $e_1=e_2$. Suppose that (e,e) is the identity in the group (M^2,Ψ) and let

$$\alpha(x) = \alpha_x = g(e,e,x), \quad \beta(x) = \beta_x = h(e,e,x) \qquad (1)$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

$$g(a, \alpha_x, \beta_x)=a, \quad h(a, \alpha_x, \beta_x)=x, \tag{3}$$
$$g(\alpha_x, \beta_x, b)=x, \quad h(\alpha_x, \beta_x, b)=b, \tag{4}$$

for arbitrary $a,b \in M$.

Dimovski [3] has proved the following lemma.

LEMMA 1. _If_ $(M,[\,])$ _is nontrivial_ (3,2) _group_, _i.e._ $|M| \geqslant 2$, _then for arbitrary_ $x,y \in M$ _it is satisfied_

$$\alpha_x \neq x, \qquad \beta_x \neq x \qquad \text{and} \qquad \alpha_x \neq \beta_y .$$

If $[\,]$ is continuous function it follows that g and h are also continuous functions and from (1) it also follows that α and β are continuous functions.

THEOREM 1. _There does not exist continuous function_ $[\,]: D^n \times D^n \times D^n \longrightarrow D^n \times D^n$ _where_ D^n _is_ n-_cube_ $(n \geqslant 1)$ _which defines a_ (3,2) _group on_ D^n.

Proof. Assume that there exists a (3,2) group with the required properties. Then α is continuous function on D^n and Brower fixed-point theorem implies that there exists a point y such that $\alpha_y = y$. This contradicts the lemma.∎

THEOREM 2. _There does not exist continuous function_ $[\,]: S^n \times S^n \times S^n \longrightarrow S^n \times S^n$ $(n \geqslant 1)$ _which defines a_ (3,2) _group on_ S^n.

Proof. Assume that there exists a continuous function $[\,]: S^n \times S^n \times S^n \longrightarrow S^n \times S^n$ which defines a (3,2) group. Since α is continuous function on S^n, it maps S^n on a compact subset of S^n. It follows from the lemma above that α is not a bijection, and since $\alpha(S^n)$ is closed subset of S^n, there exists a point $y \in S^n$ and $\varepsilon > 0$ such that $B(y,\varepsilon) \cap \alpha(S^n) = \emptyset$ where $B(y,\varepsilon) = \{z \in S^n \mid d(z,y) < \varepsilon\}$. The set $S^n \setminus B(y,\varepsilon)$ is homeomorphic to n-cube and $\alpha(S^n \setminus B(y,\varepsilon)) \subseteq S^n \setminus B(y,\varepsilon)$. Brower fixed-point theorem implies that there exists a point $z \in S^n \setminus B(y,\varepsilon)$ such that $\alpha_z = z$ and this contradicts the lemma.∎

THEOREM 3. _There does not exist continuous function_ $[\,]: M^3 \longrightarrow M^2$ _where_ M _is a connected subset of_ R, _such that_