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ON A CLASS OF VECTOR VALUED GROUPS

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Abstract. Vector valued groups are defined in [1], and some existence conditions of a kind of finite vector valued groups are given in [2]. Here we consider (2m,m)-groups and show that there is an analogy between the theory of (2m,m)-groups and the theory of binary groups.

0. In [1], (m + k,m)-groups are defined. Let $m \geqslant 1$ and $G \neq \emptyset$. $(G,[\ ])$ is a (2m,m)-group iff:

i) $[\ ]:(x_1^{2m}) \longmapsto [x_1^{2m}]$ is an associative map from G^{2m} into G^m, i.e. $[x_1^i[x_{i+1}^{2m+i}]x_{2m+i+1}^{3m}] = [[x_1^{2m}]x_{2m+1}^{3m}]$ for each $i \in \{1,2,\dots,m\}$;

and

ii) $(\forall \underline{a},\underline{b} \in G^m)(\exists \underline{x},\underline{y} \in G^m)\ [\underline{a}\ \underline{x}] = \underline{b} = [\underline{y}\ \underline{a}]$.

In i), (x_1^{2m}) stands for $(x_1,x_2,\dots,x_{2m})$ and $[x_1^{2m}]$ stands for $[x_1x_2\dots x_{2m}]$.

If we define a binary operation "$\circ$" on G^m by

(1) $$\underline{x} \circ \underline{y} = [\underline{x}\ \underline{y}]$$

then i) and ii) imply that $(G^m,\circ)$ is a group.

It is clear that a (2,1)-group is the same as a group, so, usually we assume that $m \geq 2$.

1. Let $\underline{e} = (e_1^m)$ be the identity element in a given $(2m,m)$-group $(G,[\ ])$ i.e. in $(G^m,\circ)$. Then the equalities

$$(e_2^m,e_1)\circ(e_2^m,e_1) = (e_2^m,e_1)^2 = [e_2^m\ e_1\ e_2^m\ e_1]$$
$$= [[e_2^m\ e_1^m\ e_1]e_1^m] = [e_2^m[e_1^m\ e_1\ e_1^{m-1}]e_m]$$
$$= [e_2^m\ e_1\ e_1^m] = (e_2^m,e_1)$$

imply that $(e_2^m,e_1) = (e_1^m)$, i.e. $e_2 = e_1 = e_m = e_{m-1} = \ldots = e_3 = e$. Hence, the components of $\underline{e}$ are equal, i.e.

$$\underline{e} = (\underbrace{e,\ldots,e}_{m}) = (e^m).$$

Moreover, $[x_1^{i-1}\ e^m\ x_i^m] = [[x_1^{i-1}\ e^m\ x_i^m]e^m]$

$$= [x_1^{i-1}[e^m\ x_i^m\ e^{m-i}]e^i] = [x_1^{i-1}\ x_i^m\ e^m] = (x_1^m),$$

i.e. for each $i \in \{1,2,\ldots,m\}$, $[x_1^{i-1}\ e^m\ x_i^m] = (x_1^m)$.

For each $i \in \{1,2,\ldots,m\}$ we define $\varphi_i : G^m \longrightarrow G^m$ by $\varphi_i(x_1^m) = [e^{m-i}\ x_1^m\ e^i]$. Then

$$(\varphi_i)^m(x_1^m) = [e^{m(m-1)}x_1^m\ e^{mi}] = (e^m)^{m-1}\circ(x_1^m)\circ(e^m)^i = (x_1^m).$$

So, $(\varphi_i)^m = \text{id}$ (identity), and hence φ_i is a permutation on G^m whose order is a divisor of m.

If for some $i \in \{1,2,\ldots,m-1\}$ $\varphi_i = \text{id}$, then for each $x \in G$, $(x^{m-i},e^i) = [e^m\ x^{m-i}\ e^i] = [e^{m-i}\ e^i\ x^{m-i}\ e^i]$ $= \varphi_i(e^i,x^{m-i}) = (e^i,x^{m-i})$, and so $x = e$. Thus for each $i \in \{1,2,\ldots,m-1\}$ $\varphi_i \neq \text{id}$ provided $|G| \neq 1$, i.e. G has more than one element.

2. Let $(G,\cdot)$ be a group. It is easy to check that $(G,[\ ])$ with $[\ ] : G^{2m} \longrightarrow G^m$ defined by (2) is a $(2m,m)$-group.

(2) $$[x_1^m\ y_1^m] = (x_1y_1, x_2y_2, \ldots, x_my_m)$$

Moreover, in this case, $(G^m,\circ)$ is the product

$$\underbrace{(G,\cdot)\times(G,\cdot)\times\ldots\times(G,\cdot)}_{m} \quad .$$

We call such $(2m,m)$-groups <u>trivial</u> $(2m,m)$-<u>groups</u> .

If $(G,[\ ])$ is a trivial $(2m,m)$-group, then for each $i\in\{1,\ldots,m-1\}$ $\varphi_i(x_1^m)=[e^{m-i}\ x_1^m\ e^i]$

$$=(e^{m-i},x_1^i)\circ(x_{i+1}^m,e^i)=(x_{i+1}^m,x_1^i)\,.$$

For example, if $m=4$, the order of φ_2 is 2 and the order of φ_3 is 4. In general, the order of φ_i is $m/\text{g.c.d.}(m,i)$.

3. If $(G,[\ ])$ is a $(2m,m)$-group and if we set

(3) $$[x_1^{2m}]=([x_1^{2m}]_1,[x_1^{2m}]_2,\ldots,[x_1^{2m}]_m),$$

then we get an algebra $(G;[\ ]_1,\ldots,[\ ]_m)$ with m $2m$-ary operations. This algebra satisfies the following conditions:

(i) For each $p\in\{1,2,\ldots,m\}$ and each $(x_1^{3m})\in G^{3m}$

$$[x_1^p[x_{p+1}^{2m+p}]_1\ldots[x_{p+1}^{2m+p}]_m\ x_{2m+p+1}^{3m}]_i$$
$$=[[x_1^{2m}]_1\ldots[x_1^{2m}]_m\ x_{2m+1}^{3m}]_i \quad ; \text{ and}$$

(ii) $(\forall\underline{a},\underline{b}=(b_1^m)\in G^m)(\exists\,\underline{x},\underline{y}\in G^m)(\forall i\in\{1,\ldots,m\})$

$$[\underline{a}\ \underline{x}]_i=b_i=[\underline{y}\ \underline{a}]_i \quad .$$

And conversselly, if an algebra $(G;[\ ]_1,\ldots,[\ ]_m)$ with m $2m$-ary operations satisfies the conditions (i) and (ii), then $(G,[\ ])$ is a $(2m,m)$-group with $[\ ]$ defined by (3).

In the case of a trivial $(2m,m)$-group $(G,[\ ])$, $[x_1^{2m}]_i=x_ix_{m+i}$, i.e. all of the operations $[\ ]_i$ are essentially binary and are gotten from the operation of the group $(G,\cdot)$.

PROPOSITION 1. <u>Let</u> $(G,[\ ])$ <u>be</u> <u>a</u> $(2m,m)$-<u>group</u>, <u>such that</u> <u>for</u> $i\in\{1,\ldots,m\}$ $[x_1^{2m}]_i=x_i*_ix_{m+i}$, <u>where</u>

$*_i : G^2 \longrightarrow G$ <u>is a binary operation</u>. <u>Then</u> $(G, [\ ])$ <u>is a trivial</u> $(2m,m)$-<u>group</u>.

<u>Proof</u>. It is easy to show that for each $i \in \{1,..,m\}$ $(G, *_i)$ is a group with identity element e. Next, $[x_1 [x_2^{2m+1}] x_{2m+2}^{3m}] = [[x_1^{2m}] x_{2m+1}^{3m}]$ implies that for each $i \in \{1,...,m-1\}$

$$(x_{i+1} *_i x_{m+i+1}) *_{i+1} x_{2m+i+1}$$

$$= (x_{i+1} *_{i+1} x_{m+i+1}) *_{i+1} x_{2m+i+1} .$$

Using this and the fact that $(G, *_i)$ is a group for each $i \in \{1,...,m\}$ it follows that $*_1 = *_2 = ... = *_{m-1} = *_m$. Hence, $(G, [\ ])$ is a trivial $(2m,m)$-group. ∎

REMARK. Since $[x_1^m e^m] = (x_1^m) = [e^m x_1^m]$, it follows that in every $(2m,m)$-group, $[x_1^{2m}]_i$ depends on x_i and x_{m+i}, for each $i \in \{1,...,m\}$.

Suppose that $(G, [\ ])$ is a trivial $(2m,m)$-group. Then $(G, [\ ])$ satisfies the following conditions for each $i \in \{1,...,m\}$:

(a) $[e^{i-1}\ x\ e^{m-1}\ y\ e^{m-i}]_j = e$ for $j \neq i$; and

(b) $[e^{m-i}\ x_1^m\ e^i] = (x_{i+1}^m, x_1^i)$.

PROPOSITION 2. <u>If</u> $(G, [\ ])$ <u>is a</u> $(2m,m)$-<u>group satisfying the conditions</u> (a) <u>and</u> (b), <u>then</u> $(G, [\ ])$ <u>is a trivial</u> $(2m,m)$-<u>group</u>.

<u>Proof</u>. Let $x * y = [x\ e^{m-1}\ y\ e^{m-1}]_1$. Let $(x_1^m) \in G^m$ and $(y_1^i) \in G^i$ for some $i \in \{1,...,m\}$. Then

$$[x_1^m\ y_1^i\ e^{m-i}] = [x_1^{i-1}\ x_i (x_{i+1}^m\ y_1^{i-1}\ y_i)\ e^{m-i}]$$

$$= [x_1^{i-1}\ x_i [e^{m-1}\ y_i\ x_{i+1}^m\ y_1^{i-1}\ e] e^{m-i}]$$

$$= [x_1^{i-1} [x_i\ e^{m-1}\ y_i\ e^{m-1}] e\ x_{i+1}^m\ y_1^{i-1}\ e^{m-i+1}]$$

$$= [x_1^{i-1}(x_i * y_i)\, e^{m-1}\, e\, x_{i+1}^m\, y_1^{i-1}\, e^{m-i+1}]$$
$$= [x_1^{i-1}(x_i * y_i)\, x_{i+1}^m\, y_1^{i-1}\, e^{m-i+1}]$$

implies that

$$[x_1^m\, y_1^m] = [x_1^{m-1}(x_m * y_m)\, y_1^{m-1}\, e]$$
$$= [x_1^{m-2}(x_{m-1} * y_{m-1})(x_m * y_m)\, y_1^{m-2}\, e^2]$$
$$= \dots = [(x_1 * y_1)(x_2 * y_2)\dots(x_m * y_m)\, e^m]$$
$$= (x_1 * y_1, x_2 * y_2, \dots, x_m * y_m)\,.$$

This shows that $(G,[\ ])$ is a trivial $(2m,m)$-group. ∎

4. Let $(G,[\ ])$ and $(K,[\ ])$ be $(2m,m)$-groups. A map $f : G \longrightarrow K$ is called $(2m,m)$-homomorphism if

$$f^{(m)}([x_1^{2m}]) = [f(x_1)\, f(x_2) \dots f(x_{2m})]\,,$$

where $f^{(m)} : G^m \longrightarrow K^m$ is the m^{th} product of f, i.e. $f^{(m)}(y_1^m) = (f(y_1), f(y_2), \dots, f(y_m))$. It is clear that f is a $(2m,m)$-homomorphism iff $f^{(m)} : (G^m, \circ) \longrightarrow (K^m, \circ)$ is a group homomorphism.

Let $f : (G^m,[\ ]) \longrightarrow (K^m,[\ ])$ be a $(2m,m)$-homomorphism, (e^m) the identity in $(G,[\ ])$, (k^m) the identity in $(K,[\ ])$ and $H = \ker(f) = \{x \mid x \in G,\ f(x) = k\} = f^{-1}(k)$. Let us examine some properties of H. First of all, H^m is a normal subgroup of $(G^m, \circ)$. Moreover, H satisfies the following conditions for each $i \in \{1,2,\dots,m\}$:

(4) $$[x_1^{i-1}\, H^m\, x_i^m] = [x_1^m\, H^m] \quad ; \quad \text{and}$$

(5) $$[x_1^m\, H^m] = [y_1^m\, H^m] \iff [(x_i)^m\, H^m] = [(y_i)^m\, H^m]\,.$$

Above, $[x_1^{i-1}\, H^m\, x_i^m]$ stands for the set $\{[x_1^{i-1}\, h_1^m\, x_i^m] \mid (h_1^m) \in H^m\}$.

For $m = 1$, the condition (5) is trivial, and the condition (4) is equivalent to H being a normal subgroup, provided that H is a subgroup.

Let us show (4). Because $e \in H$, it follows that $[e^i\, H^m\, e^{m-i}] = H^m$ for each $i \in \{0,1,\dots,m\}$. Since H^m is normal in $(G^m,\circ)$ it follows that $[x_1^m\, H^m] = [H^m\, x_1^m]$. Now, $[x_1^{i-1} H^m x_i^m] = [x_1^{i-1}\, H^m\, x_i^m\, e^m] = [x_1^{m-1} H^m (x_i^m\, e^{i-1}) e^{m-i+1}]$
$= [x_1^{i-1}\, x_i^m\, e^{i-1}\, H^m\, e^{m-i+1}] = [x_1^m\, H^m]$.
This shows that (4) follows only from the fact that H^m is a normal subgroup of $(G^m,\circ)$.

The condition (5) is a consequence of the following equivalences:

$$[x_1^m\, H^m] = [y_1^m\, H^m] \Longleftrightarrow f^{(m)}(x_1^m) = f^{(m)}(y_1^m)$$
$$\Longleftrightarrow f(x_i) = f(y_i) \quad \text{for each } i \in \{1,\dots,m\}$$
$$\Longleftrightarrow f^{(m)}((x_i)^m) = f^{(m)}((y_i)^m) \quad \text{for each } i \in \{1,..,m\}$$
$$\Longleftrightarrow [(x_i)^m\, H^m] = [(y_i)^m\, H^m] \quad \text{for each } i \in \{1,\dots,m\}.$$

We say that a subset H of a given (2m,m)-group (G,[ ]) is a (2m,m)-<u>subgroup</u> if H^m is a subgroup of $(G^m,\circ)$. A (2m,m)-subgroup H of (G,[ ]) is called <u>normal</u> (2m,m)-<u>subgroup</u> if it satisfies the condition (5) and H^m is a normal subgroup of $(G^m,\circ)$.

Hence ker(f) is a normal (2m,m)-subgroup of a given (2m,m)-group (G,[ ]) for any (2m,m)-homomorphism f from (G,[ ]) to some (2m,m)-group (K,[ ]).

5. Let (H,[ ]) be a normal (2m,m)-subgroup of (G,[ ]). We define a relation $\sim$ on G by

(6) $\quad a \sim b \Longleftrightarrow [a^m\, H^m] = [b^m\, H^m]$.

It is easy to check that $\sim$ is an equivalence on G. We denote the factor set $G/\sim$ by G/H, and its elements by aH. Next we define [ ] on G/H by:

(7) $\quad [(x_1H)(x_2H)\dots(x_{2m}H)] = ([x_1^{2m}]_1H,\dots,[x_1^{2m}]_mH)$.

PROPOSITION 3. (i) $(G/H,[\ ])$ <u>is a</u> $(2m,m)$-<u>group</u>.
(ii) <u>The natural map</u> $\pi : G \longrightarrow G/H$ <u>defined by</u> $\pi(x) = xH$ <u>is a</u> $(2m,m)$-<u>homomorphism</u>.
(iii) $\ker(\pi) = H$.

<u>Proof</u>. (i) Suppose that $x_jH = y_jH$ for each $j \in \{1,2,\dots,2m\}$, i.e. $[(x_j)^m H^m] = [(y_j)^m H^m]$. Then (5) implies that $[x_1^m H^m] = [y_1^m H^m]$ and $[x_{m+1}^{2m} H^m] = [y_{m+1}^{2m} H^m]$. Now, $[[x_1^{2m}] H^m] = [x_1^m[x_{m+1}^{2m} H^m]]$
$= [x_1^m[y_{m+1}^{2m} H^m]] = [x_1^m H^m y_{m+1}^{2m}] = [y_1^m H^m y_{m+1}^{2m}] = [[y_1^{2m}] H^m]$.
This, and (5) imply that for each $i \in \{1,\dots,m\}$
$[x_1^{2m}]_iH = [y_1^{2m}]_iH$, i.e. $[\ ]$ is well defined.

The associativity and the condition <u>0</u>. ii) for $[\ ]:(G/H)^{2m} \longrightarrow (G/H)^m$ follow directly from the associativity and the condition <u>0</u>. ii) for $[\ ]:G^{2m} \longrightarrow G^m$.

(ii) $\pi^{(m)}([x_1^{2m}]) = \pi^{(m)}([x_1^{2m}]_1,\dots,[x_1^{2m}]_m)$
$= ([x_1^{2m}]_1H, \dots ,[x_1^{2m}]_mH) = [x_1H\dots\dots x_{2m}H]$
$= [\pi(x_1)\ \pi(x_2)\dots\pi(x_{2m})]$.

(iii) $\ker(\pi) = \{x \mid \pi(x) = eH\} = \{x \mid xH = eH\}$
$= \{x \mid x \in H\} = H$. ∎

The $(2m,m)$-group $(G/H,[\ ])$ is called $(2m,m)$-<u>factor group</u> of G by H .

PROPOSITION 4. <u>Let</u> $(H,[\ ])$ <u>be a normal</u> $(2m,m)$-<u>subgroup of a given</u> $(2m,m)$-<u>group</u> $(G,[\ ])$. <u>Then</u> $(G^m/H^m,\circ)$ <u>is isomorphic to the group</u> $((G/H)^m,\circ)$ <u>via an isomorphism</u> g <u>defined by</u> $g((x_1^m)H^m) = (x_1H,\dots,x_mH) = \pi^{(m)}((x_1^m))$.

<u>Proof</u>. g is well defined because $[x_1^m H^m] = [y_1^m H^m]$ implies that $\pi^{(m)}((x_1^m)) = \pi^{(m)}((y_1^m))$. Since $\pi^{(m)}$ is an

epimorphism it follows that g is an epimorphism . If $g((x_1^m)H^m) = (eH)^m$, then $\pi^{(m)}((x_1^m)) = (eH)^m$, which implies that $[(x_1^m)H^m] = H^m$. Hence, g is a monomorphism. ∎

6. Suppose that $(G,[\ ])$ is a trivial $(2m,m)$-group gotten from a group $(G,\cdot)$. Let H be a normal subgroup of $(G,\cdot)$. Then H^m is a normal subgroup of $(G^m,\circ)$. To show that H satisfies (5), let $(x_1^m),(y_1^m) \in G^m$. Then

$[x_1^m\ H^m] = [y_1^m\ H^m] \iff x_iH = y_iH$ for each $i \in \{1,\dots,m\}$

$\iff [(x_i)^m\ H^m] = [(y_i)^m\ H^m]$ for each $i \in \{1,\dots,m\}$.

Hence, $(H,[\ ])$ is a normal $(2m,m)$-subgroup of $(G,[\ ])$.

Conversely, suppose that H is a normal $(2m,m)$-subgroup of a trivial $(2m,m)$-group $(G,[\ ])$. If $h_1,h_2 \in H$, then $[h_1\ e^{m-1}\ h_2\ e^{m-1}] = (h_1h_2,e^{m-1}) \in H^m$, and $(h_1,e^{m-1})^{-1} = (h_1^{-1},e^{m-1}) \in H^m$. Hence, H is a subgroup of $(G,\cdot)$. Because H^m is a normal subgroup of $(G^m,\circ)$, it follows that $(x,e^{m-1})H^m = H^m(x,e^{m-1})$ i.e. $xH = Hx$ for each $x \in G$. Hence, H is a normal subgroup of $(G,\cdot)$.

The above discussion shows that the notion of normal $(2m,m)$-subgroups makes sense only for "pure" $(2m,m)$-groups, i.e. for $(2m,m)$-groups that are not trivial $(2m,m)$-groups. Otherwise, it is the same as the notion of normal subgroups.

7. A $(2m,m)$-group can be thought of as an algebra $(G,e;\{[\ ]_i, [\backslash]_i, [/]_i\}_{i=1,\dots,m})$ where $[\ ]_i, [\backslash]_i, [/]_i$ are $2m$-ary operations , e is a constant, and the following identities are satisfied for each $i \in \{1,\dots,m\}$:

$$[x_1^p[x_{p+1}^{p+2m}]_1 \dots [x_{p+1}^{p+2m}]_m\ x_{p+2m+1}^{3m}]_i =$$

$$= [[x_1^{2m}]_1 \ldots [x_1^{2m}]_m \, x_{2m+1}^{3m}]_i \, ,$$

$$[x_{m+1} \ldots x_{2m} \setminus x_1 \ldots x_m]_i = x_i \, ,$$

$$[x_1 \ldots x_m \, / \, x_{m+1} \ldots x_{2m}]_i = x_i \, , \quad \text{and}$$

$$[e^m \, x_1^m]_i = x_i = [x_1^m \, e^m]_i .$$

Hence, the class of (2m,m)-groups is a variety of algebras. So, for better understanding of the (2m,m)-groups it is needed to obtain canonical forms for the elements in free (2m,m)-groups.

We note that free (2m,m)- groups are not trivial (2m,m)-groups.

References

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