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# SEMILATTICES OF SIMPLE n-SEMIGROUPS

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The purpose of this paper is to show that the well known characteristic of semilattices of simple semigroups ([1],[2]) could be generalized for the class of n-semigroups for $n>2$.

## 1. SOME DEFINITIONS AND RESULTS

Let S be on n-semigroup, i.e. an algebra with an associattive n-ary operation $(x_1,x_2,\ldots,x_n)\to x_1x_2\ldots x_n$. An n-semigroup S is called a semilattice if S is commutative, idempotent and satissies the sollowing identity

$$x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k} = x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k},$$

where $i_1+i_2+\ldots+i_k=j_1+j_2+\ldots j_k=n$, $i_\nu, j_\nu>0$.

A congruence on an n-semigroup S is called a semilattice congruence if $S/\alpha$ is a n-semilattice.

A nonempty subset A of an n-semigroup S is called an ideal of S iff $a\in S$, $x_i\in S$ imply $x_1\ldots x_{i-1}\,a\,x_i\ldots x_n\in A$ for every $i=1,2,\ldots,n$.

An ideal J of S is said to be completely prime iff $x_1x_2\ldots x_n\in J$ $x_1\in J$ or $x_2\in J$ or ... or $x_n\in J$.

A subset F of S is a filter in S iff $J=S\setminus F$ is a completely prime ideal.

An ideal A of an n-semigroup S is completely semiprime if for any $x\in S$, $x^n\in A$ implies $x\in A$.

An characterisation of all semilattice decompositions of an n-semigroup S in terms of completely prime ideals is given in [3]. The least semilattice congruence is denated by $\eta$. The minimal filtre in S which contains x is denoted by N(x), i.e. N(x) is the filtre generated by x. The classes of the congruence $\eta$ are called N-classes. If $x\in S$, then the N-class which contains x is denated by $N_x$. The class $N_x$ is the largest n-subsemigroup of S containing x and containing no proper completely prime ideals.

An n-semigroup S is said to be *$\eta$-simple* iff S has no proper completely prime ideals. For n-ary case the following theorem is given in [3] (3.5):

*1.1 Is* I *is an ideal of some* N *class of an* n-*semigroup* S, *then* I *has no proper completely prime ideal*.

As a consequence of 1.1 we conclude that:

*1.2 Every* n-*semigroup is a semilattice of* $\eta$ - *simple* n-*semigroups*.

The principal left, right two sided ideals and ideal of a semigroup S generated by an element $x\in S$ have the following form:

$$L(x) = x\cup S^{n-1}x, \quad R(x) = x\cup xS^{n-1},$$

$$I(x) = x\cup S^{n-1}x\cup xS^{n-1}\cup S^{n-1}xS^{n-1},$$

$$J(x) = x\cup S^{n-1}x\cup S^{n-2}xS\cup\ldots\cup xS^{n-1}\cup S^{n-1}xS^{n-1}.$$

An n-semigroup S is *left* (right) *simple* if S is its only left (right) ideal; S is *two-sided simple* if S is its only two-sided ideal; S is *simple* if S is its only ideal. These notions can be characterised in the following way:

*1.3 Let* S *be an* n-*semigroup*:

S *is left simple iff* $S^{n-1}a=S$ *for all* $a\in S$;

S *is two-sided simple iff* $S^{n-1}aS^{n-1} = S$ *for all* $a\in S$

S *is simple iff* $S=(\bigcup_{i=2}^{n-1} S^{n-i}aS^{i-1})\cup S^{n-1}aS^{n-1}$ *for all* $a\in S$.

We note also the following results.

*1.4 A semilattice* S *with respect to the relation* $\leqslant$ *defined by*

$$x\leqslant y \Leftrightarrow xy^{n-1}=x$$

is partial ordered set.

## 2. A SEMIGROUP AND ITS N-CLASSES

Now we shalt establish some equivalent statements on the N-classes, when they are left simple, and certain properties of S in terms of either elements of S or some tipes of ideals of S. ([2], II.4.9 for the binary case).

2.1 The following conditions on an n-semigroup S are equivalent.

i) Every $\eta$-class is a left simple n-semigroup.

ii) Every left ideal of S is completely semiprime and ideal

iii) For every $x\in S$, $x\in S^{n-1}x^n$ and $xS^{n-1}\subseteq S^{n-1}x$.

iv) For every $x\in S$, $N_x=L_x$.

v) For every $x\in S$, $N_x=\{y\in S \mid x\in S^{n-1}y_1x\in S^{n-1}x\}$

vi) Every left ideal is a union of $\eta$-classes.

Proof. i) $\Rightarrow$ ii) Let L be a left ideal. If $x^n\in L$, then $x^n\in L\cap N_x$; hence $L\cap N_x$ is a left ideal of $N_x$ and we must have $L\cap N_x=N_x$. But then $x\in L$ and thus L is completely semiprime. If $x\in L$ and $y_1,y_2,\dots,y_{n-1}\in S$, then $y_1y_2\dots y_{n-1}x\in$ $\in L\cap N_{y_1y_2\dots y_{n-1}}x$. Hence $L\cap N_{y_1y_2\dots y_{n-1}}x$ is a left ideal of $N_{y_1y_2\dots y_{n-1}x}$ and we have that $L\cap N_{y_1y_2\dots y_{n-1}}=N_{y_1y_2\dots y_{i-1}xy_i\dots y_{n-1}}$ for every $i=1,2,\dots,n-1$. But then $y_1y_2\dots y_{i-1}xy_i\dots y_{n-1}\in N_{y_1y_2\dots y_{n-1}x}$ for every $i=1,2,\dots,n-1$. This implies $y_1\dots y_{i-1}xy_i\dots y_{n-1}\in L$, which means that L is an ideal of S.

ii) $\Rightarrow$ iii) For any $x\in S$, $S^{n-1}x^n$ is a left ideal of S and thus it is completely semiprime. Since $x^{2n-1}\in S^{n-1}x^n$, we have $x\in S^{n-1}x^n\subseteq S^{n-1}x$. The set $S^{n-1}x$ is a left ideal and thus an ideal of S and contains x, so that $xS^{n-1}\subseteq J(x)\subseteq S^{n-1}x$.

iii) $\Rightarrow$ iv) First we will prove that $L_x\subseteq N_x$. By the hypothesis, $x\in S^{n-1}x^n\subseteq S^{n-1}x$. Then $L(x)=S^{n-1}x$ for every $x\in S$. If $y\in L_x$, then $L(x)=L(y)$ and thus $x=a_1a_2\dots a_{n-1}y$, $y=b_1b_2\dots b_{n-1}x$ for some $a_1,a_2,\dots,a_{n-1}$, $b_1,b_2,\dots,b_{n-1}\in S$. Therefore $N_x=N_{a_1a_2\dots a_{n-1}y}=N_{xy^{n-1}}=N_{yx^{n-1}}=N_{b_1b_2\dots b_{n-1}x}=N_y$ and thus $y\in N_x$, that is $L_x\subseteq N_x$.

Now we will prove that the relatation $\mathcal{L}$, defined by $x\mathcal{L}y \iff L(x)=L(y)$ is a semilattice congruence. Since $\eta$ is the least semilattice congruence we have that $N_x\subseteq L_x$.

By the hypothesis we have that $u\in S^{n-1}u^n=L(u^n)$. Thus $L(u)\subseteq L(u^n)$, $L(u^n)=S^{n-1}u^n\subseteq S^{n-1}u=L(u)$, i.e. $L(u)=L(u^n)$.

We show next that for any $x_1,x_2,\dots,x_n\in S$,

$$L(x_1x_2\dots x_n)=L(x_1)\cap L(x_2)\cap\dots\cap L(x_n) \qquad (1)$$

Since $(x_1x_2\dots x_n)^n\in x_1x_2\dots x_{n-1}\dots x_1x_2\dots x_{n-1}S^{n-1}\subseteq S^{n-1}x_1x_2\dots x_{n-1}\dots x_1x_2\dots x_{n-1}\subseteq S^{n-1}x_nx_1x_2\dots x_{n-1}=L(x_nx_1x_2\dots x_{n-1})$, we have that $x_1x_2\dots$

$x_n \in L(x_n x_1 \ldots x_{n-1})$ i.e. $L(x_1 x_2 \ldots x_{n-1} x_n) = L(x_n x_1 \ldots x_{n-1})$.

Similarly

$L(x_n x_1 \ldots . x_{n-1}) \subseteq L(x_{n-1} x_n x_1 \ldots x_{n-2})$ and so

$L(x_1 x_2 \ldots x_n) \subsetneq L(x_n x_1 \ldots x_{n-1}) \subseteq \ldots \subsetneq L(x_1 x_2 \ldots x_n)$. Thus

$L(x_1 x_2 \ldots x_n) \subseteq L(x_1) \cap L(x_2) \cap \ldots \cap L(x_n)$.

Let $z \in L(x_1) \cap L(x_2) \cap \ldots \cap L(x_n)$, then $z = a_{11} a_{12} \ldots a_{1n-1} x_1$, $z = a_{21} a_{22} \ldots a_{2n} x_2, \ldots, z = a_{n1} a_{n2} \ldots . a_{nn-1} x_n$, for some $a_{ij} \in S$, where $i = 1,2,\ldots,n$; $j = 1,2,\ldots, n-1$ and consequently

$$Z^n \in L(a_{11} a_{12} \ldots a_{1n-1} x_1 \ldots . a_{n1} a_{n2} \ldots a_{nn-1} x_n) \subseteq$$

$$\subseteq L(x_n x_1 a_{21} a_{22} \ldots a_{2n-1} x_2 \ldots . a_{n-1n-1} x_{n-1}) \subseteq \ldots \subseteq L(x_1 x_2 \ldots x_n)$$

From the equality (1) follows that

$$L_{x_{i-1} x_{i_2} \ldots x_{i_n}} = L_{x_{j_1} x_{j_2} \ldots x_{j_n}}, \quad L_{x_1^{i_1} x_2^{i_2} \ldots . x_k^{i_k}} = L_{x_1^{j_1} x_2^{j_2} \ldots x_k^{jk}},$$

where $(i_1, i_2, \ldots, i_m)$, $(j_1, j_2, \ldots, j_m)$ are some permutation of the numbres $(1,2,\ldots,n)$ and $i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k = n$.

iv) $\Rightarrow$ v) Let x be any element of S. Since $x^n \in N_x$, then $x^n \in L_x$. But $L_x = \{y \in S \mid L(x) = L(y)\}$. So, we obtain $L(x) = L(x^n)$. From this it follows that $x \in L(x^n) = x^n \cup S^{n-1} x^n$. If $x = x^n$, then $x = x^n \in S^{n-1} x^n \subsetneq S^{n-1} x$. If $x \in S^{n-1} x^n$, we have that $x \in S^{n-1} x$. Thus $L(x) = S^{n-1}$. Then we can write

$$N_x = L_x = \{y \in S \mid L(x) = L(y)\} = \{y \in S \mid y \in S^{n-1} x,\ x \in S^{n-1} y\}.$$

v) $\Rightarrow$ v) If L is a left ideal of S, x an element of L, and y an element of $N_x$, then $y \in S^{n-1} x \subseteq L$, that is vi) holds

vi) $\Rightarrow$ i) It suffices to show that $N_x \subseteq N_x^{n-1} y$ for all $y \in N_x$. For $y, z \in N_x$, the hypothesis implies $N_x \subseteq L(y^{2n-1})$. Since $z \in N_x \subseteq L(y^{2n-1}) = y^{2n-1} \cup S^{n-1} y^{2n-1}$ we have that $z = a_1 \ldots . a_{n-1} y^{2n-1}$ for some $a_1, a_2, \ldots, a_{n-1} \in S$. Hence $N_x = N_z = N_{a_1 \ldots a_{n-1}} y^{2n-1} = N_{a_1 a_2 \ldots a_{n-1}}$ and $a_1 \ldots a_{n-1} y \in N_x$ which implies $z = a_1 \ldots . a_{n-1} y^{2n-1} = a_1 \ldots . a_{n-1} y y^{2n-3} y \subseteq N_x^{n-1} y$, and this proves that $N_x \subseteq N_x^{n-1} y$.

A similar proposition holds for right simple N-classes.

By a simple modification of the proof of 2.1, one can prove the following theorem:

2.2. The following conditions on an n-semigroup S are equivalent

i) Every class is two-sided simple.

ii) Every two-sided ideal of S is completely semiprime and ideal.

iii) For every $x \in S$, $x \in S^{n-1}x^nS^{n-1}$

iv) For every $x \in S$, $N_x = I_x$.

v) For every $x \in S$, $N_x = \{y \in S \mid x \in S^{n-1}yS^{n-1},\ y \in S^{n-1}xS^{n-1}\}$

vi) Every two-sided ideal is union of $\eta$-classes.

## 3. $Y_s$ IS LINEARLY ORDERED

In this section we perform an analysis simplar to that of section two. Here we suppose that $Y_s$ is linearly ordered, where $Y_s = S/\eta$ is the set of $\eta$-classes of S which constitutes the greatest semilattice decomposition of S.

3.1 The following conditions on an n-semigroup are equivalent.

i) Every $\eta$-class is left simple and $Y_s$ is linearly ordered.

ii) Every left ideal of S is completely prime and ideal.

iii) For every $x_1, x_2, \ldots, x_n \in S$, $\{x_1, x_2, \ldots, x_n\} \cap S^{n-1}x_1x_2\ldots x_n \neq \emptyset$ and $xS^{n-1} \subseteq S^{n-1}x$.

Proof. i) $\Rightarrow$ ii) Let L be a left ideal of S. Since every N-class is left simple, by 2.1, L is a union of N-classes. If $x_1x_2\ldots x_n \in L$, then $N_{x_1x_2\ldots x_n} \subseteq L$. By hypothesis $Y_s$ is linearly ordered, which means that $N_{x_{i_1}} \leqslant N_{i_2} \leqslant \ldots \leqslant N_{x_{i_n}}$, where $(i_1, i_2, \ldots, i_n)$ is some permutation of the numbers $(1,2,\ldots,n)$. We have that

$$N_{x_{i_1}} = N_{x_{i_1}^{n-1}x_{i_2}} = N_{x_i^{n-1}x_{i_2}^{n-1}x_{i_3}} = \ldots N_{x_{i_1}^{n-1}x_{i_2}^{n-1}\ldots x_{i_{n-1}}^{n-1}x_{i_n}} = N_{x_i^{n-1}x_{i_2}^{n-1}\ldots}$$

$$_{\ldots x_{i_{n-1}}^{n-1}x_{i_n}^{n}} = N_{y_{i_1}^{n-1}x_{i_n}x_{i_2}^{n-1}x_{i_n}\ldots x_{i_{n-1}}^{n-1}x_{i_n}} = N_{x_{i_1}x_{i_2}\ldots x_{i_n}},$$

and thus L is completely prme.

Let $x_1, x_2, \dots, x_{n-1} \in S$ and $y \in L$, then $x_1x_2\dots x_n y \in N_{x_1x_2\dots x_{n-1}} \subseteq L$. Since $N_{x_1x_2\dots x_{n-1}y} = N_{x_1x_2\dots x_{i-1}yx_i\dots x_n}$, we have that $x_1x_2\dots x_{i-1}yx_i\dots x_n \in L$ and thus L is ideal of S.

ii) ⇒ iii) For any $x_1, x_2, \dots, x_n \in S$, $S^{n-1}x_1x_2\dots x_n$ is a left ideal of S and completely prime ideal. Since $(x_1x_2\dots x_n)^n \in S^{n-1}x_1x_2\dots x_n$, we have that $x_1x_2\dots x_n$ $S^{n-1}x_1x_2\dots x_n$ and thus either $x_1 \in S^{n-1}x_1x_2\dots x_n$ or $x_2 \in S^{n-1}x_1x_2\dots x_n$ or ... or $x_n \in S^{n-1}x_1x_2\dots x_n$. From 2.1 it foollows that $xS^{n-1} \subseteq S^{n-1}x$.

iii) ⇒ i) Let $x, y \in S$ and suppose that $x \in S^{n-1}xy^{n-1}$; the case $y \in S^{n-1}xy^{n-1}$ is treated similarly. Then $x = a_1a_2\dots a_{n-1}xy^{n-1}$ for some $a_1, a_2, \dots, a_{n-1} \in S$, and thus $N_x = N_{a_1a_2\dots a_{n-1}xy^{n-1}} = N_{a_1a_2\dots a_{n-1}xy^{n-2}y} = N_{a_1a_2\dots a_{n-1}xy^{n-2}y^n} = N_{a_1a_2\dots a_{n-1}xy^{n-1}y^{n-1}} = N_{xy^{n-1}}$, that is $N_x \leqslant N_y$ and therefore $Y_s$ is linearly ordered. Left simplicity of each $N_x$ follows immediately form 2.1 since $x \in S^{n-1}x^n$ for all $x \in S$.

A proof of the next theorem can be given by a modification of the proof of 3.1.

3.2. *The following conditions on an n-semigroup S are quivalent.*

i) *Every $\eta$ - class is two-sided simple and $Y_s$ is linearly ordered*

ii) *Every ideal of S is completely prime and ideal*

iii) *For every* $x_1, x_2, \dots, x_n \in S$, $\{x_1, x_2, \dots, x_n\} \cap S^{n-1}x_1x_2\dots x_nS^{n-1} \neq \emptyset$.

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