SOME PROPER QUASIVARIETIES OF SUBALGEBRAS OF SEMIGROUPS

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Abstract. The well known Cohn-Rebane's Theorem ([1]) states that every universal algebra $\mathcal{A} = \langle A;\Omega\rangle$ can be ambadded in a semigroup $\mathcal{S} = \langle S;\cdot\rangle$ in such a way that the operations of the algebra $\mathcal{A}$ are restrictions of inner left translations in the semigroup $\mathcal{S}$. Then we say that $\mathcal{A}$ is a subalgebra of $\mathcal{S}$, the concept being introduced by Kuroš ([7]). If $\mathcal{K}$ is a variety of semigroups then the class of Ω-algebras (algebras in a signature Ω) that are subalgebras of semigroups in $\mathcal{K}$, denoted by $\mathcal{K}(\Omega)$, is a quasivariety ([8] , pg. 274) and need not to be variety ([2], [3]). In this article we find a necessary condition for a set of varieties of semigroups K such that $\mathcal{K}(\Omega)$ is a proper quasivariety for every $\mathcal{K}$ K and Ω containing at least two non-constant operators, not both unar ones, generalizing a part of the results obtained in [4] and [5].

1. PRELIMINARIES

Say that an Ω-algebra $\mathcal{A} = \langle A;\Omega\rangle$ is a subalgebra of a semigroup $\mathcal{S} = \langle S;\cdot\rangle$ if $A \subseteq S$ and there is a mapping $\psi:\Omega \to S$ such that for every n-ary operator ω belonging to Ω and for every $a_1,\dots,a_n \in A$

$$\omega(a_1,\dots,a_n) = \psi(\omega)a_1\cdot\ldots\cdot a_n \qquad (1)$$

The set of n-ary operators belonging to an arbitrary signature Ω will be denoted by $\Omega(n)$ $(n=0,1,\dots)$.

Let $\mathcal{K}$ be a variety of semigroups and Ω an arbitrary signature. As we have already mentioned, the problem is to determine whether $\mathcal{K}(\Omega)$ is a variety and that is independent of the constants in Ω. Thus we can suppose that $\Omega(0) = \emptyset$.

Before passing on we give necessary denotations and definitions.

Let u stand for a word in an arbitrary alphabeth. Then denote by: c(u) - the set of symbols occuring in u (content of u); d(u) - the number of apperances of symbols in u (lenght of u); (i)u - the i-th symbol from left to right, occuring in u; u(i) - the i-th symbol from right to left, occuring in u.

Let ϕ be an Ω-formula, i.e. a formula in a first order language defined by a signature Ω. Let f be a mapping from the set of operators occuring in ϕ in to the set of variables not occuring in ϕ. If we submit every occurence of an operator ω in ϕ by the variable $f(\omega)$, we obtain a sequence of variables and eventually logical symbols and brackets. This sequence can be easyly interpreted as a formula in the multiplicative operator signature, with respect to a class of semigroups. The last formula is called *SEM-instance* of the formula ϕ with respect to the mapping f and is denoted by ϕ^S. If we submit the word „term" instead of „formula" we shall get the definition of a *SEM-instance* of a term with respect to the mapping f.

The following lemma is, in a sence, natural and expectable. We shall omit the proof for the sake of compactness and shortness of the exposition.

LEMMA 1.1. ([6]) *Let $\mathbb{K}$ be a class of semigroups and Ω be a signature. An open Ω-formula ϕ is valid in the class $\mathbb{K}(\Omega)$ if and only if any SEM-instance of ϕ is valid in $\mathbb{K}$.* ::

2. RESULTS

We shall take in to a consideration the varieties of semigroups that are axiomatizable by regular identities (*regular varieties*), by ends-preseving identities (*ends-preserving varieties*) and by balanced identities (*balanced varieties*), whereas an identity u=v is *regular* iff (if and only if) $c(u)=c(v)$, it is *ends-preserving* iff $u(1)=v(1)$ and $(1)u=(1)v$ and it is *balanced* iff the numbers of appearances of any variable in u and v are equal.

For a class of semigroups $\mathbb{K}$ and an arbitrary signature Ω denote by $V\mathbb{K}(\Omega)$ the variety of Ω-algebras defined by the all identities valid in $\mathbb{K}(\Omega)$. Obviously, $\mathbb{K}(\Omega)$ is a variety iff $\mathbb{K}(\Omega) = V\mathbb{K}(\Omega)$.

LEMMA 2.1. *Let $\mathbb{K}$ be a variety of semigroups axiomatizable by regular identities and Ω,Ω' be two signatures such that $\Omega\subseteq\Omega'$. Then every algebra belonging to $V\mathbb{K}(\Omega)$ is a subalgebra of an Ω-restriction of an Ω'-algebra belonging to $V\mathbb{K}(\Omega')$.*

Proof. Let $\mathbb{A} = \langle A;\Omega\rangle$ belong to $V\mathbb{K}(\Omega)$ and $a\notin A$. Define an Ω'-algebra $\mathbb{A}' = \langle A\cup\{a\};\Omega'\rangle$ by

$$\omega_{\mathbb{A}'}(a_1,\ldots,a_n) = \begin{cases} \omega_{\mathbb{A}}(a_1,\ldots,a_n) & \text{if } \omega\in\Omega,\ a_1,a_2,\ldots,a_n\in A \\ a & \text{otherwise,} \end{cases}$$

whereas $a_1,\ldots,a_n$ are arbitrary elements in $A\cup\{a\}$.

Let ϕ be an identity valid in $VK(\Omega')$. Thus ϕ is valid in $K(\Omega')$ and according to Lemma 1.1., every SEM-instance ϕ^S of ϕ is valid in K, which implies regularity of ϕ^S. We conclude that ϕ is a regular Ω'-identity, i.e. ϕ is of the tipe u=v whereas c(u)=c(v).

If ϕ is an Ω-formula then the fact that ϕ^S is valid in K implies that ϕ is valid in $K(\Omega)$. Thereby ϕ is valid in A and moreover ϕ is valid in A'.

Otherwise, both the terms u and v are equal to a in A'. ::

As a consequence of the previous lemma we have this useful theorem:

<u>THEOREM 2.2</u>. <u>Let</u> K <u>be a variety of semigroups axiomatizable by regular identities</u>, Ω, Ω' <u>be two signatures and</u> $\Omega \subsetneq \Omega'$. <u>If</u> $K(\Omega')$ <u>is a variety then</u> $K(\Omega)$ <u>is a variety</u>.

<u>Proof</u>. Let $K(\Omega')$ be a variety and $A = \langle A;\Omega\rangle$ belong to the variety $VK(\Omega)$. By Lemma 2.1. A is a subalgebra of a restriction of an Ω'-algebra A' belonging to $VK(\Omega')$. Thus $A' \in K(\Omega')$ and if A' is a subalgebra of a semigroup S, $S \in K$, then A is a subalgebra of S too. Thereby, $A \in K(\Omega)$ and $VK(\Omega) = K(\Omega)$. ::

Now we turn on the ends-preserving varieties of semigroups exposing a necessary condition for the class $K(\Omega)$ to be a proper quasivariety!

<u>THEOREM 2.3</u>. <u>Let</u> K <u>be an ends-preserving variety of semigroups and</u> $\Omega = \{\omega, \tau\}$ <u>be a signature</u> ($\omega \in \Omega(n)$, $\tau \in \Omega(m)$, $\omega \neq \tau$). <u>If there exist</u> Ω-<u>terms</u> t_1, t_2, t_3, t_4 <u>and a variable-word</u> w <u>such that</u>

i) $(1)t_1 = \omega$, $(1)t_2 = \tau$, $t_1(1) \neq w(1) \neq t_2(1)$

ii) <u>if</u> u_i <u>is a</u> SEM-<u>instance of</u> t_i <u>with respect to a mapping</u> f $(i=1,2,3,4)$ <u>then</u> $u_1 w = u_3$ <u>and</u> $u_2 w = u_4$ <u>are identities valid in</u> K,

<u>then</u> $K(\Omega)$ <u>is a proper quasivariety</u>.

<u>Proof</u>. Consider the Ω-quasiidentity ϕ: $t_1 = t_2 \rightarrow t_3 = t_4$. It is valid in $K(\Omega)$ because its SEM-instance $u_1 = u_2 \rightarrow u_3 = u_4$ is valid in K (Lemma 1.1.). It remains to prove that ϕ is not a consequence of the identities valid in $K(\Omega)$, i.e. to find an Ω-algebra A belonging to $VK(\Omega)$ and not satisfying the quasiidentity ϕ.

Let $A = \langle A;\Omega\rangle$ be the algebra generated in $VK(\Omega)$ by the set $\{a_1, a_2\}$ $(a_1 \neq a_2)$ and with one defining relation between the generators: $t_1(a_1,a_2) = t_2(a_1,a_2)$, whereas $t_i(a_1,a_2)$ is the „continued product", i.e. it is obtained from t_i by substituting every occurence of the variable w(1) in t_i by a_1 and the occurencesof the other variables by a_2 $(i=1,2,3,4)$. Now it is enough to prove that $t_3(a_1,a_2)$ is not equal in A to $t_4(a_1,a_2)$.

First notice that $(1)t_3=\omega$, $(1)t_4=\tau$ and that both $t_3(a_1,a_2)$ and $t_4(a_1,a_2)$ end on the element a_1 (utilize Lemma 1.1., the fact that $\mathbb{K}$ is ends-preserving and the condition ii) of this theorem).

Suppose that $t_3(a_1,a_2) = t_4(a_1,a_2)$ in $\mathbb{A}$. That means that there exists a sequence of continued products $v_1(a_1,a_2),\ldots,v_k(a_1,a_2)$ such that v_1 (v_k) graphically coincides with t_3 (t_4 resp.) and: $v_i=v_{i+1}$ is identity valid in $\mathbb{K}(\Omega)$ or there are subwords $v_i'(a_1,a_2)$ and $v_{i+1}'(a_1,a_2)$ of $v_i(a_1,a_2)$ and $v_{i+1}(a_1,a_2)$ respectively such that $v_i'(a_1,a_2) = v_{i+1}'(a_1,a_2)$ is exactly the defining relation $t_1(a_1,a_2) = t_2(a_1,a_2)$ $(i=1,2,\ldots,k-1)$. Because of the ends-preserving identities in $\mathbb{K}$ and Lemma 1.1. the second possibility must be applied for some minimal j $(1 \le j < k)$. If $v_j'(a_1,a_2)$ is a proper subword of $v_j(a_1,a_2)$ then the ends of $v_j(a_1,a_2)$ are equal to those of $v_{j+1}(a_1,a_2)$. Thus, for some minimal $j_0 \ge j$, $v_{j_0}(a_1,a_2)$ coincides with $t_1(a_1,a_2)$. But this is imposible because they end on diferent elements. ::

An atempt to utilize the last theorem leads us to the following corollary.

COROLLARY 2.4. Let $\Omega = \{\omega,\tau\}$ be a signature containing two diferent operators, not both unar ones. Let $\mathbb{K}$ be an ends-preserving variety of semigroups such that there exists an identity of the following type, valid in $\mathbb{K}$:

a) $u = v$, whereas $d(u) \ne d(v)$

b) nonbalanced identity (specialy, nonregular identity)

c) $x_1\cdot\ldots\cdot x_k x_{k+1}\cdot\ldots\cdot x_q = x_1\cdot\ldots\cdot x_k x_{i_1}\cdot\ldots\cdot x_{i_s} x_q$, $k \ge 1$, $i_1 \notin \{1,2,\ldots,k+1\}$, $i_1,i_2,\ldots,i_s \le q$.

Then $\mathbb{K}(\Omega)$ is a proper quasivariety.

Commentary. If $\mathbb{K}$ is a regular variety then an analog assertion is valid for every signature containing at least two operators, not both unar ones (Theorem 2.2. and this corollary).

Proof. We take over the notation of Theorem 2.3. . Thus we are to find the terms t_1,t_2,t_3 and t_4 and the variable-word w. Let u_1,u_2,u_3 and u_4 be SEM-instance of t_1,t_2,t_3 and t_4 respectively, with respect to the mapping $f:\omega \to x$, $\tau \to y$. Let $\omega \in \Omega(n)$, $\tau \in \Omega(m)$. We can suppose that $n \ge 2$.

a) Let u and v be $x_{i_1}\cdot\ldots\cdot x_{i_p}$ and $x_{j_1}\cdot\ldots\cdot x_{j_q}$ respectively, $p < q$ and $x_k, x \notin c(u) \cup c(v)$. Choose t_1,t_2,t_3,t_4 and w as follows:

$_{\omega}r_{\tau}r\, x_1^{r(n+m-2)+1-q}x_{j_1}\cdot\ldots\cdot x_{j_q}$, $_{\tau}r_{\omega}r\, x_1^{r(n+m-2)+1-q}x_{j_1}\cdot\ldots\cdot x_{j_q}$,

$_{\omega}r_{\tau}r\, x_1^{r(n+m-2)+1-q}x_{i_1}\cdot\ldots\cdot x_{i_p}\, {}_{\omega}s\, x_k^{s(n-1)+q-p}$,

$_{\tau}r_{\omega}r\, x_1^{r(n+m-2)+1-q}x_{i_1}\cdot\ldots\cdot x_{i_p}\, {}_{\omega}s\, x_k^{s(n-1)+q-p}$ and $x^s x_k^{s(n-1)+q-p}$ resp. .

It is obvious that the conditions of Theorem 2.3. are fulfilled.

b) If there is a nonbalanced identity valid in K then obviously, there is an identity $u=v$ valid in K such that $d(u)\neq d(v)$. So we are back on the case a).

c) If $x_{i_1} \notin c(x_{k+2}\cdot\ldots\cdot x_q)$ then we are on the case b). Otherwise, let w be obtained from $x_{k+2}\cdot\ldots\cdot x_q x_i^p$ $(i>q)$ by substituting every occurence of x_{i_1} by x_ν, $\nu>i$. Take t_1, t_2, t_3 and t_4 as follows:

$_{\omega}r_{\tau}r\, x_1^{r(n+m-2)-k}x_1x_2\cdot\ldots\cdot x_{k+1}$, $_{\tau}r_{\omega}r\, x_1^{r(n+m-2)-k}x_1x_2\cdot\ldots\cdot x_{k+1}$,

$_{\omega}r_{\tau}r\, x_1^{r(n+m-2)-k}x_1x_2\cdot\ldots\cdot x_k u x_i^p$, $_{\tau}r_{\omega}r\, x_1^{r(n+m-2)-k}x_1x_2\cdot\ldots\cdot x_k u x_i^p$,

whereas u is obtained from $x_{i_1}\cdot\ldots\cdot x_{i_s}$ by substituting the variable x_{i_1} by ω^q and p is a nonnegative integer such that the last two words are terms. Now apply Theorem 2.3. . ::

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