Smile MARKOVSKI

n-SUBGROUPOIDS OF COMMUTATIVE GROUPOIDS

Dedicated to Prof. Blagoj S. Popov

A description of the classes of n-groupoids which can be embedded in corresponding ways into commutative groupoids is given in this paper. It is shown that these classes are varieties of n-groupoids, and axiom systems for these varieties are obtained.

1. *Binary terms and commutativity.* Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a set of variables, and let $*$ be a binary operator symbol. Define the set T of $*$-terms (or, briefly, of terms) as the minimal subset of the set of all finite strings on $X \cup \{*\}$, which satisfies the properties

(i) $x \in T$, for every $x \in X$,

(ii) if $u, v \in T$, then $*uv \in T$.

Let $t, t_1, t_2, \in T$. Then t_1 is said to be a *subterm* of t_2 iff t_1 is a substring of t_2. We denote by $t(y_1, \ldots, y_k)$ that the set of variables which occur in the term t is $\{y_1, \ldots, y_k\}$.

Let $t(y_1, \ldots, y_k) \in T$ and $a_1, \ldots, a_k$ be elements of a set A. Then by $t_{y_1, \ldots, y_k}[a_1, \ldots, a_k]$ we denote the string on the set $A \cup \{*\}$ obtained from t in such a way that every occurrence of a variable y_i in t is changed by a_i, for $i = 1, 2, \ldots, k$. In that case we say that $t_{y_1, \ldots, y_k}[a_1, \ldots, a_k]$ is an A-word, which is an *instance* of t. (Sometimes, when the set of variables $\{y_1, \ldots, y_k\}$ is known, we write simply $t[a_1, \ldots, a_k]$ instead of $t_{y_1, \ldots, y_k}[a_1, \ldots a_k]$.) Denote the set of all A-words by $W(A)$. It can easily be proved that $T = W(X)$ and $A \subseteq W(A)$. An A-word u is said to be a *subword* of an A-word v iff u is a substring of v.

Consider a groupoid $\mathbf{A} = (A; \circ)$ and an A-word u. We denote by $u_{\mathbf{A}}$ the "value" of u in A, i.e. the product in $\mathbf{A}$ obtained from u when every occurrence of $*$ in u is replaced by $\circ$.

By the commutative law we mean the law $*xy = *yx$. If $u,v \in T$ and if as a consequence of the commutative law we have an identity $u = v$, then we denote it by $u \underset{c}{=} v$. A term $t = *t_1t_2$ is said to be *commutatively invariant* iff $t_1 \underset{c}{=} t_2$. For a term t, define $C(t) \underset{c}{=} \{t' \mid t = t', t' \in T\}$. Using an induction on the number of occurrences of the sign $*$ in a term, one can prove

1.1. Let a term t contain r subterms of forms $*t_1 t_2$, such that s of them be commutatively invariant. Then $|C(t)| = 2^{r-s}$.

1.2. $t_1, t_2, t_3, t_4 \in T \Rightarrow (*t_1 t_2 \in C(*t_3 t_4) \Leftrightarrow t_1 \in C(t_3), t_2 \in C(t_4)$ or $t_1 \in C(t_4), t_2 \in C(t_3))$.

As a consequence of **1.1** and **1.2** we can give the following description of the free commutative groupoid $\mathbf{F}_A$, generated by a set A. Let $u \in W(A)$ and define $C(u) = \{v \mid u \underset{c}{=} v, v \in W(A)\}$. Then **1.1** and **1.2** are true when words are regarded instead of terms. Now, let $F_A = \{C(u) \mid u \in W(A)\}$ and define an operation $\cdot$ on F_A by

$$\cdot\, C(u)\, C(v) = C(*uv).$$

Then it follows from **1.2** that $\cdot$ is well defined, and $\mathbf{F}_A = (F_A; \cdot)$.

2. *t-subgroupoids of commutative groupoids.* A universal algebra $\mathbf{A} = (A; f)$ with one n-ary operation f is said to be an *n-groupoid*. (We assume that $n \geqslant 2$.)

Let $t(x_1, \ldots, x_n)$ be a $*$-term (with n distinct variables). An n-groupoid $\mathbf{A} = (A; f)$ is said to be a *t-subgroupoid* of a groupoid $\mathbf{G} = (G; \circ)$ iff $A \subseteq G$ and for every $a_1, \ldots, a_n \in A$

$$f_{\mathbf{A}}(a_1, \ldots, a_n) = t[a_1, \ldots, a_n]_G. \tag{2.1}$$

The principal result of this is

THEOREM 2.1. Let $\mathbf{A} = (A; f)$ be an n-groupoid and $t(x_1, \ldots, x_n)$ be a term. Then $\mathbf{A}$ is a t-subgroupoid of a commutative groupoid iff $\mathbf{A}$ satisfies all the identities

$$f(x_1, \ldots, x_n) = f(x_{i_1}, \ldots, x_{i_n}), \tag{2.2}$$

where $\nu \mapsto i_\nu$ is a permutation of the set $\{1, 2, \ldots, n\}$ such that $t(x_1, \ldots, x_n) \underset{c}{=} t_{x_1, \ldots, x_n}[x_{i_1}, \ldots, x_{i_n}]$.

Proof. If $\mathbf{A}$ is a t-subgroupoid of a commutative groupoid, then it is clear that $\mathbf{A}$ satisfies the identities (2.2).

Now, suppose that $\mathbf{A}$ satisfies all the identities (2.2). Let $\mathbf{F}_A$ be the free commutative groupoid generated by the carrier A of the n-groupoid $\mathbf{A}$.

An element $C(u) \in F_A$, where $u \in W(A)$, is said to be *reduced* iff each $v \in C(u)$ does not contain a subword w such that w is an instance of t. Denote by R the set of reduced elements of F_A, and define a binary operation $\circ$ on R as follows:

If

$$C(u), C(v), \quad \cdot C(u)\, C(v) = C(*uv) \in R,$$

then

$$\circ C(u)\, C(v) = C(*uv).$$

If $C(u), C(v) \in R$, $C(*uv) \notin R$, then there is $w \in C(*uv)$ such that w has a subword which is an instance of t. But, as a consequence of 1.2. it follows that w itself is an instance of t. Thus, $w = t\,[a_1, \ldots, a_n]$ for some $a_1 \ldots, a_n \in A$, and in this case we put

$$\circ C(u)\, C(v) = C(a) = \{a\},$$

where $a = f_{\mathbf{A}}(a_1, \ldots, a_n)$.

The operation $\circ$ is well defined. Namely, if $w_1, w_2 \in C(*uv)$ and w_1, w_2 are instances of t, then we have $w_1 = t\,[a_1, \ldots, a_n]$, $w_2 = t\,[a_{i_1}, \ldots, a_{i_n}]$ where $a_1, \ldots, a_n \in A$ and $\nu \mapsto i_\nu$ is a permutation of $\{1, 2, \ldots n\}$. Since $w_1 \underset{C}{=} w_2$, it follows by (2.2) that $f_{\mathbf{A}}(a_1, \ldots, a_n) = f_{\mathbf{A}}(a_{i_1}, \ldots, a_{i_n})$.

It is clear that the groupoid $\mathbf{R} = (R; \circ)$ is commutative.

We can suppose that $A \subseteq R$, identifying $C(a)$ and a, for $a \in A$. Also, the equation (2.1) is satisfied for the groupoid $\mathbf{R}$:

$$f_{\mathbf{A}}(a_1, \ldots, a_n) = C(f_{\mathbf{A}}(a_1, \ldots, a_n)) = C(t[a_1, \ldots, a_n]) =$$
$$= t\,[a_1, \ldots, a_n]_{\mathbf{R}},$$

for every $a_1, \ldots, a_n \in A$.

This completes the proof that $\mathbf{A}$ is a t-subgroupoid of the commutative groupoid $\mathbf{R}$.

We note that the above Theorem is a generalization of a result of G. Čupona's [1]. Further to this, we can give a generalization of another definition and result of that paper.

An n-groupoid $\mathbf{A} = (A; f)$ is said to be *commutative* iff it satisfies the equations

$$f_{\mathbf{A}}(a_1, \ldots, a_n) = f_{\mathbf{A}}(a_{i_1}, \ldots, a_{i_n})$$

for every permutation $\nu \mapsto i_\nu$ of the set $\{1, 2, \ldots, n\}$, and every $a_1, \ldots, a_n \in A$.

Let $t(x_1, \ldots, x_n)$ be a $*$-term with n distinct variables ($n \geqslant 2$). A groupoid $(G; \circ)$ is said to be *t-commutative* iff the n-groupoid $(G; t)$ is commutative.

THEOREM 2.2 The class of t-subgroupoids of t-commutative groupoids and the class of commutative n-groupoids are equal.

Proof. It is clear that every t-subgroupoid of a t-commutative groupoid is a commutative n-groupoid.

Let $\mathbf{A}=(A;f)$ be a commutative n-groupoid and let $S=\{u\in W(A)\mid u$ has no subword which is an instance of $t\}$. Define an operation o on S as follows:

If

$$u, v, *uv \in S,$$

then

$$\circ uv = *uv.$$

If $u, v, \in S$, $*uv \notin S$, then $*uv = t\,[a_1, \ldots, a_n]$ for some $a_1, \ldots, a_n \in A$, and we put in this case

$$\circ uv = f_{\mathbf{A}}(a_1, \ldots, a_n).$$

In such a way we get a groupoid $\mathbf{S}=(S;\circ)$, and $A\subseteq S$.

Define a congruence β on S as follows: Let $u_1, \ldots, u_n \in S$ and $\nu \to i_\nu$ be a permutation of the set $\{1, 2, \ldots, n\}$. Then, we put

$$t\,[u_1, \ldots, u_n]\,\alpha\, t\,[u_{i_1}, \ldots, u_{i_n}],$$

and β is the minimal congruence generated by α.

The quotient groupoid $\mathbf{G}=\mathbf{S}/\beta$ is t-commutative.

Note that for $a\in A$, $u_1, \ldots, u_n \in S$ we have $a\,\beta\, t\,[u_1, \ldots, u_n]$ iff $u_1, \ldots, u_n \in A$, and in that case $a = f_{\mathbf{A}}(u_1, \ldots, u_n)$. It follows that if $a, b \in A$ and $a\,\beta\, b$, then $a=b$, i.e. we can suppose that $A\subseteq G$. Also, since $f_{\mathbf{A}}(a_1, \ldots, a_n)\,\beta\, t\,[a_1 \ldots, a_n]$ for $a_1, \ldots, a_n \in A$, we have that $\mathbf{A}$ is a t-subgroupoid of $\mathbf{G}$ as well.

REFERENCES

[1] Čupona, G.: *On n-groupoids*, Mat. Bilt. 1 (XXVII), 1978, Skopje (5—11).

[2] Markovski, S.: *n-Subgroupoids of cancellative groupoids*, Год. збор. Мат. фак. 32, 1981, Скопје (45—51).

Смиле МАРКОВСКИ

n-ПОДГРУПОИДИ ОД КОМУТАТИВНИ ГРУПОИДИ

Резиме

Се дава опис на класата *n*-групоиди што можат да се сместат во комутативни групоиди така што операциите на *n*-групоидите се рестрикции од соодветни полиномни групоидни операции. Се покажува, имено, дека за секоја полиномна групоидна операција соодветната класа *n*-групоиди е многукратност.

University "Kiril i Metodij",
Faculty of Mahematic Sciences,
Skopje — Yugoslavia.