

SUBALGEBRAS OF SEMILATTICES

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Let $\mathcal{A}=(A, \Omega)$ be an algebra, S a semigroup and $\omega \mapsto \bar{\omega}$ a mapping of Ω into S , such that $A \subseteq S$ and

$$(1) \quad \omega(a_1, \dots, a_n) = \bar{\omega} a_1 \dots a_n$$

for every n -ary operator $\omega \in \Omega$ and every $a_1, \dots, a_n \in A$. $\mathcal{A}=(A, \Omega)$ is called a *subalgebra* of the semigroup S .

It is well known ([1] p. 185) that every algebra is a subalgebra of some semigroup. Subalgebras of commutative semigroups are characterized in [2].

In this paper we describe the class of algebras which are subalgebras of semilattices.

THEOREM. *An algebra $\mathcal{A}=(A, \Omega)$ is a subalgebra of some semilattice if and only if the following condition is satisfied:*

(*) *For every pair of terms t_1 and t_2 , with the same sets of symbols, $t_1=t_2$ is an identity in \mathcal{A} .*

Proof. It is obvious that the condition (*) is satisfied in every subalgebra of a semilattice.

Suppose that the condition (*) is satisfied in the algebra $\mathcal{A}=(A, \Omega)$. We are going to prove that this algebra is a subalgebra of a semilattice. We can assume that:

- (i) A and Ω are disjoint sets;
- (ii) Different operators from Ω define different operations in A ;
- (iii) Ω does not contain 0-ary operators, so that the operators, i.e. elements of Ω , are operations in A .

Let $K=A \cup \Omega$ and let \mathcal{M} be the family of finite subsets of the set K (including the empty set). \mathcal{M} is a semilattice with the set theoretical union as an operation. If we put $x=\{x\}$, for every $x \in K$, we have $K \subseteq \mathcal{M}$, so that \mathcal{M} may be considered as a free semilattice with an identity on the base K .

Define in \mathcal{M} a relation of „neighbourhood“ in the following way:
If $S, T \in \mathcal{M}$ and

$$S = S' \cup a, \quad T = S' \cup \{\omega, a_1, \dots, a_n\},$$

where $a = \omega(a_1, \dots, a_n)$ in the algebra \mathcal{A} , we say that (S, T) and (T, S) are two pairs of neighbours generated by the operation ω .

If there exists a sequence $S_0, S_1, \dots, S_p \in \mathcal{M}$ such that $S = S_0, T = S_p, p \geq 0$ and (S_{i-1}, S_i) is a pair of neighbours for every $i \in \{1, 2, \dots, p\}$, we say that S and T are equivalent and denote this $S \sim T$.

The relation \sim is a congruence on the semilattice \mathcal{M} and

$$(2) \quad a = \omega(a_1, \dots, a_n) \text{ (in } \mathcal{A}) \Rightarrow a \sim \{\omega, a_1, \dots, a_n\}.$$

We are going to prove the following implication:

$$(3) \quad a, b \in \mathcal{A} \Rightarrow (a \sim b \Rightarrow a = b),$$

from which, because of (2), we come to the conclusion that the algebra \mathcal{A} is a subalgebra of the semilattice $\mathcal{P} = \mathcal{M} / \sim$.

To every element $S \in \mathcal{M}$ we correspond its „value“ $[S]$ in the following way:
If one of the sets $S_\Omega = S \cap \Omega, S_A = S \cap A$ is empty, we put $[S] = S$.

Let

$$S_\Omega = \{\omega_1, \dots, \omega_r\}, \quad S_A = \{a_1, \dots, a_s\}$$

be non-empty sets. If all the operations from S_Ω are unary we put

$$[S] = \{b_1, \dots, b_s\},$$

where

$$b_v = \omega_1 \dots \omega_r(a_v), \quad v = 1, \dots, s.$$

Finally, suppose that at least one of the operations from S_Ω for instance ω_r , is not unary. There exist positive integers i, j such that $\omega_1 \omega_2 \dots \omega_r^i(a_1^j, a_2, \dots, a_s)$ is a „continued product“ and if a is the value of this product we put $[S] = a$. From the condition (*), it follows that $[S]$ does not depend on the quadruple ω_r, a_1, i, j .

Further on we denote $[S_1 \cup S_2 \cup \dots \cup S_k]$ with $[S_1, S_2, \dots, S_k]$.

The following lemmas are simple consequences from the condition (*) and the definition of the transformation [].

LEMMA 1. $[[S]] = [S]$.

LEMMA 2. If $S_A = T_A = \emptyset$, then

$$[S, T, U] = [S, [T, U]].$$

LEMMA 3. If $a = \omega(a_1, \dots, a_n)$ in \mathcal{A} and $[S, a] \in \mathcal{A}$ then

$$[S, a] = [S, \omega, a_1, \dots, a_n].$$

We are going to prove the following

LEMMA 4. Let $a = S_0, S_1, \dots, S_p$ be a sequence of elements from \mathcal{M} , where $a \in \mathcal{A}$, and (S_i, S_{i+1}) is a pair of neighbours for every $i \in \{0, 1, \dots, p-1\}$. If ω_i is the generating operation for the pair (S_i, S_{i+1}) then

$$[\omega_v, a] = [\omega_1, \dots, \omega_p, S_v] = a$$

for every $v \in \{1, \dots, p\}$.

Proof. If $p=1$ then we have

$$S_1 = \{\omega_1, a_1, \dots, a_n\} \text{ or } S_1 = \{a, \omega_1, a_1, \dots, a_n\}$$

where $\omega_1(a_1, \dots, a_n) = a$. Then

$$[\omega_1, a] = \omega_1(a^n) = \omega_1((\omega_1(a_1, \dots, a_n))^n) = \omega_1(a_1, \dots, a_n) = a$$

$$[\omega_1, S_1] = \omega_1(a^{n-1}, \omega_1(a_1, \dots, a_n)) = \omega_1(a^n) = a, n \geq 2$$

and

$$[\omega_1, S_1] = [\omega_1, \omega_1, a_1] = \omega_1^2(a_1) = \omega_1(a) = a, \text{ for } n=1.$$

Assume that the assertion of lemma 4. for the sequence $a = S_0, S_1, \dots, S_q$ holds. We have two possible cases:

$$(I) S_{q+1} = S' \cup \{\omega, a_1, \dots, a_n\}, S_q = S' \cup b, b = \omega(a_1, \dots, a_n)$$

$$(II) S_{q+1} = S' \cup b, S_q = S' \cup \{\omega, a_1, \dots, a_n\}, b = \omega(a_1, \dots, a_n).$$

For case (I) we have

$$\begin{aligned} [\omega_1, \dots, \omega_q, \omega, S_{q+1}] &= [\omega_1, \dots, \omega_q, \omega, \omega, S', a_1, \dots, a_n] = \\ &= [\omega_1, \dots, \omega_q, S', \omega, a_1, \dots, a_n]. \end{aligned}$$

From lemma 3. and an inductive assumption it follows that

$$\begin{aligned} a = [\omega_1, \dots, \omega_q, S_q] &= [\omega_1, \dots, \omega_q, S', b] = [\omega_1, \dots, \omega_q, S', \omega, a_1, \dots, a_n] = \\ &= [\omega_1, \dots, \omega_q, \omega, S_{q+1}]. \end{aligned}$$

From lemma 2. we have

$$[\omega, a] = [\omega, [\omega_1, \dots, \omega_q, \omega, S_{q+1}]] = [\omega_1, \dots, \omega_q, \omega, S_{q+1}] = a,$$

$$[\omega, \omega_1, \dots, \omega_q, S_v] = [\omega, [\omega_1, \dots, \omega_q, S_v]] = [\omega, a] = a$$

for $v=1, \dots, q$.

Consider case (II).

$\omega \in \{\omega_1, \dots, \omega_q\}$ so it follows that

$$[\omega, \omega_1, \dots, \omega_q, S_{q+1}] = [\omega_1, \dots, \omega_q, S_{q+1}].$$

By the inductive assumption we also have

$$[\omega_1, \dots, \omega_q, S_q] = a.$$

If in the set $\{\omega_1, \dots, \omega_q\}$ there is a non-unary operation, then $[\omega_1, \dots, \omega_q, S_{q+1}] \in \mathcal{A}$, and from lemma 3. we have

$$\begin{aligned} [\omega_1, \dots, \omega_q, S_{q+1}] &= [\omega_1, \dots, \omega_q, S, b] = \\ &= [\omega_1, \dots, \omega_q, S^2, \omega, a_1, \dots, a_n] = [\omega_1, \dots, \omega_q, S_q] = a. \end{aligned}$$

If all the operations $\omega_1, \dots, \omega_q$ are unary then in can be easily seen that $[\omega_1, \dots, \omega_q, S_q] = a$ implies that $[\omega_1, \dots, \omega_q, S_{q+1}] = a$

Q.E.D.

Now, implications (3) can easily be verified. If $a, b \in \mathcal{A}$ and $a \sim b$, then there exists a sequence $a = S_0, S_1, \dots, S_p = b$ such that (S_{i-1}, S_i) is a pair of neighbours for every $i \in \{1, \dots, p\}$. If $p = 0$ then $a = S_0 = b$. Let $p \geq 1$ and the pair (S_{i-1}, S_i) be generated by the operation ω_i . From lemma 4. it follows that

$$[\omega_1, a] = [\omega_2, a] = \dots = [\omega_p, a] = [\omega_1, \dots, \omega_p, b] = a.$$

Consider the sequence $b = S_p, \dots, S_1, S_0 = a$. We have

$$b = [\omega_1, \dots, \omega_p, a] = [\omega_1, \dots, \omega_{p-1}, [\omega_p, a]] = [\omega_1, \dots, \omega_{p-1}, a] = [\omega_1, a] = a.$$

By this we finish the proof of the theorem.

Notes.

1. If Ω does not contain unary operators and to every n -ary operation $\omega \in \Omega$ we correspond a binary operation ω' by $\omega'(x, y) = \omega(x^{n-1}, y)$, then we have the algebra (\mathcal{A}, Ω') with binary operations, satisfying the condition (*) if and only if the algebra (\mathcal{A}, Ω) satisfies the same condition. We also have that

$$\omega(x_1, \dots, x_n) = \omega'^{n-1}(x_1, \dots, x_n).$$

2. We call the algebra (\mathcal{A}, Ω) Ω -semilattice if there exists the semilattice P such that $\mathcal{A} \subseteq P$ and

$$(1') \quad \omega(a_1, \dots, a_n) = a_1 \dots a_n$$

for every n -ary operation $\omega \in \Omega$ and $a_1, \dots, a_n \in \mathcal{A}$. It can be easily verified that (\mathcal{A}, Ω) is Ω -semilattice if and only if the following condition is satisfied:

(*) For every pair of terms t_1, t_2 , with the same sets of variables $t_1 = t_2$ is an identity in (\mathcal{A}, Ω) .

Obviously, the condition (*) is necessary.

If (*) is satisfied in (\mathcal{A}, Ω) then all the operators, of the same arity n , define equal n -ary operations, so we may take that for every n there is at most one operation in Ω . If there exists a unary operation $\omega \in \Omega$ we have that $\omega(x) = x$ for every $x \in \mathcal{A}$. Let ω be an n -ary operation in Ω ($n \geq 2$), and a binary operation „ \cdot ” be defined by $x \cdot y = \omega(x^{n-1}, y)$. Then we obtain a semilattice (\mathcal{A}, \cdot) such that

$$\rho(x_1, \dots, x_m) = x_1 \dots x_m$$

for every m -ary operation $\rho \in \Omega$.

3. An algebra \mathcal{A} satisfies condition (*) if and only if all the identities of the following forms hold in \mathcal{A} :

$$3.1 \quad \omega x_1 \dots x_n = \omega x_{i_1} x_{i_2} \dots x_{i_n};$$

$$3.2 \quad \rho x_1 \dots x_m = \rho \omega x_1 \dots x_m;$$

$$3.3 \quad \omega \rho x_1 \dots x_m = \omega x_1 \dots x_{i-1} \rho x_i \dots x_m;$$

$$3.4 \quad \omega_1^{r_1} \dots \omega_p^{r_p} x_1^{\alpha_1} \dots x_q^{\alpha_q} = \omega_1^{s_1} \dots \omega_p^{s_p} x_1^{\beta_1} \dots x_q^{\beta_q}.$$

\mathcal{A} satisfies (*) if it satisfies all the identities 3.1, 3.2, 3.3 and 3.4', where

$$3.4' \quad \omega_1 \dots \omega_r x_1^{\alpha_1} \dots x_q^{\alpha_q} = \rho_1 \dots \rho_s x_1^{\beta_1} \dots x_q^{\beta_q}.$$

($\omega, \rho, \omega_v, \rho_v$ are arbitrary elements of Ω ; i_1, \dots, i_n is a permutation of $1, \dots, n$ and $n, m, r_v, s_v, \alpha_v, \beta_v, r, s$ are positive integers such that both the hand sides of the corresponding identities are Ω -terms).

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PODALGEBRE POLUMREŽE

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REZIME

Neka je $\mathcal{A}=(A, \Omega)$ algebra, S polugrupa i $\omega \mapsto \bar{\omega}$ preslikavanje Ω u S tako da je $A \subseteq S$ i

$$\omega(a_1, \dots, a_n) = \bar{\omega}a_1, \dots, a_n$$

za svaki n -arni operator $\omega \in \Omega$ i $a_1, \dots, a_n \in A$. Kažemo da je \mathcal{A} podalgebra polugrupe S .

U radu je dat opis klase podalgebri polumreža. Važi sledeća

TEOREMA. Algebra $\mathcal{A}=(A, \Omega)$ je podalgebra neke polumreže ako i samo ako zadovoljava sledeći uslov:

(*) Za proizvoljne terme t_1 i t_2 , sa jednakim skupovima simbola u algebri \mathcal{A} je zadovoljen identitet $t_1 = t_2$.