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ON QUASIVARIETIES OF GENERALIZED SUBALGEBRAS

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Throughout this paper we shall use the usual notations and notions of the theory of models, assuming that all languages are first order languages without relational symbols different from ≡, where ≡ denotes the equality symbol in the given language. The aim of this paper is to give a generalization of Theorem (M), by using a generalization of Theorem (L).

<u>THEOREM (M)</u> ([2], p. 274). Let $L_1 \subseteq L_2$ be two languages and let $\Sigma_1(\Sigma_2)$ be a set of L_1-quasiidentities (L_2-quasiidentities). Then the class $\mathbb{K}$ of all L_1-algebras $\mathcal{A} \in \underline{\text{Mod}}\Sigma_1$, for which there exists an L_2-algebra $\mathcal{B} \in \underline{\text{Mod}}\Sigma_2$ such that $\mathcal{B}$ is an L_2-extension of $\mathcal{A}$, is a quasivariety. □

<u>THEOREM (L)</u> ([1]). Let $L_1 \subseteq L_2$ be two languages and let Σ be a set of L_2-formulas, let $\mathcal{A}$ be an L_1-algebra. Then there exists an L_2-extension $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \in \underline{\text{Mod}}\Sigma$ iff $\mathcal{A}$ is a model of the set of all open L_1-formulas, which are theorems in the theory defined by Σ. □

Here by a quasiidentity we mean a formula of the form $\neg\phi_1 \vee \cdots \vee \neg\phi_k \vee \phi$, where $\phi, \phi_1, \ldots, \phi_k$ $(k \geq 0)$ are atomic formulas (or identities), i.e. formulas of the form $\xi \equiv \eta$, where $\xi, \eta \in \mathrm{Term}(L)$. A quasivariety is a class $\mathbb{K}$ of L-algebras such that $\mathbb{K} = \underline{\mathrm{Mod}}\,\Sigma$, where Σ is a set of quasiidentities.

1. Generalized subalgebras. Let L_1 and L_2 be two languages such that $L_1 \cap L_2$ does not contain operational symbols. (This assumption is only for technical reasons, as can be seen from what follows.) For each n-ary operational symbol $f \in L_1$ let a term $f^\wedge \in \mathrm{Term}(L_2)$ be given such that $f^\wedge = f^\wedge(x_1,\dots,x_n)$ contains no more than n distinct variables. Then an L_1-algebra $\mathcal{A}$ is said to be a $^\wedge$-subalgebra (or generalized subalgebra) of an L_2-algebra $\mathcal{B}$ iff $|\mathcal{A}| \subseteq |\mathcal{B}|$ and

$$f_{\mathcal{A}}(a_1,\dots,a_n) = f^\wedge_{\mathcal{B}}(a_1,\dots,a_n) \qquad (1.1$$

for all $a_1,\dots,a_n \in |\mathcal{A}|$, $f \in L_1$ ($|f| = n$, $n = 0,1,2,\dots$).

Let $\mathcal{A}$ be an L_1-algebra. We are asking under what circumstances there exists an L_2-algebra $\mathcal{B}$ belonging to a class of L_2-algebras $\mathbb{K}$, such that $\mathcal{A}$ is a generalized subalgebra of $\mathcal{B}$ (for a given $^\wedge$). We shall give here an answer which is a generalization of Theorem (L).

Let the languages L_1, L_2 and the mapping $^\wedge$ be as above. Let Σ be a set of L_2-formulas and

$$\Sigma' = \Sigma \cup \{fx_1\dots x_n \equiv f^\wedge(x_1,\dots,x_n) \mid f \in L_1\}.$$

THEOREM 1 ([5], [7]). An L_1-algebra $\mathcal{A}$ is a $^\wedge$-subalgebra of some L_2-algebra $\mathcal{B} \in \underline{\mathrm{Mod}}\,\Sigma$ iff $\mathcal{A}$ satisfies all open formulas, which are theorems in the theory defined by the set of formulas Σ'.

Proof. Let $\mathcal{A}$ be a $^\wedge$-subalgebra of an L_2-algebra $\mathcal{B} \in \underline{\mathrm{Mod}}\,\Sigma$. Form an expansion $\mathcal{C}$ of $\mathcal{B}$ for the language $L_1 \cup L_2$ by putting $f \in L_1 \Rightarrow f_{\mathcal{C}}(b_1,\dots,b_n) = f^\wedge_{\mathcal{B}}(b_1,\dots,b_n$

for all $b_1,\dots,b_n \in |\mathcal{B}|$, $f \in L_1$ ($|f| = n$, $n = 0,1,2,\dots$). Then $\mathcal{A}$ is subalgebra of $\mathcal{C}|L_1$, $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, and so every open L_1-formula, satisfied by $\mathcal{C}$, is satisfied by $\mathcal{A}$ too.

Conversely, suppose that $\mathcal{A}$ satisfies all open L_1-formulas which are theorems in the theory defined by Σ'. Then, by Theorem (L), there exists an $L_1 \cup L_2$-algebra $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, such that $\mathcal{A}$ is an L_1-subalgebra of $\mathcal{C}$. Consider the resfriction $\mathcal{B}|L_2$. Since $\mathcal{C} \in \underline{\text{Mod}}\,\Sigma'$, we have $\mathcal{B} \in \underline{\text{Mod}}\,\Sigma$ and, furthermore, for all $f \in L_1$, $a_1,\dots,a_n \in |\mathcal{A}|$,

$$f_{\mathcal{A}}(a_1,\dots,a_n) = f_{\mathcal{C}}(a_1,\dots,a_n)$$

(as $\mathcal{A}$ is an L_1-subalgebra of $\mathcal{C}$),

$$f_{\mathcal{C}}(a_1,\dots,a_n) = \hat{f}_{\mathcal{B}}(a_1,\dots,a_n)$$

(as $\mathcal{B} = \mathcal{C}|L_2$)

So, (1.1) is satisfied. □

2. Malcev's theorem for generalized subalgebras. Now we shall give a generalization of Malcev's theorem in the case when generalized subalgebras are considered. Since the usual notion of subalgebra can be obtained as a special kind of generalized subalgebra, the proof given below is another proof of Theorem (M) too (and more elementary, in author's opinion).

THEOREM 2. Let L_1, L_2 be two languages and ^ be a given mapping (defined as above), and let Σ_1 (Σ_2) be a set of L_1-quasiidentities (L_2-quasiidentities). Then the class $\mathbb{K}$, consisting of all L_1-algebras $\mathcal{A} \in \underline{\text{Mod}}\,\Sigma_1$ which are ^-subalgebras of algebras belonging to the $\underline{\text{Mod}}\,\Sigma_2$, is a quasuvariety.

Proof. Denote by Σ_o the set of all open L_1-formulas as in the proof of Theorem 1. It suffices to prove that $\underline{\text{Mod}}\,\Sigma_o = \underline{\text{Mod}}\,\Sigma_o'$, where Σ_o' consists of all quasiidentities from Σ_o.

We may assume that Σ_o contains only formulas of the following form:

$$\neg\phi_1 \vee \dots \vee \neg\phi_n \quad (2.1)$$

$$\phi_1 \vee \dots \vee \phi_m \quad (2.2)$$

$$\neg\phi_1 \vee \dots \vee \neg\phi_k \vee \phi_{k+1} \vee \dots \vee \phi_{k+p} \quad (m,n,k,p > 0) \quad (2.3)$$

where ϕ_i are atomic formulas (identities).

Let $\Sigma_3 = \Sigma_1 \cup \Sigma_2 \cup \{fx_1 \dots x_n \equiv f^\wedge(x_1,\dots,x_n) \mid f \in L_1\}$. Then Σ_3 is a set of quasiidentities, and

$$\mathbb{M} = (\underline{\text{Mod}}\,\Sigma_3)\big|_{L_1} \subseteq \underline{\text{Mod}}\,\Sigma_o.$$

Since each one element L_1-algebra belongs to $\mathbb{M}$, we get that Σ_o does not contain formulas of the forms (2.1). The free $L_1 \cup L_2$-algebras belong to the class $\underline{\text{Mod}}\Sigma_3$, and so every formula of the form (2.2) is equivalent to a formula of the form ϕ_i, for some $i: 1 \le i \le m$. As $\underline{\text{Mod}}\Sigma_3$ is closed under direct products, we have that every formula of the form (2.3) is equivalent to a formula of the form $\neg\phi_1 \vee \dots \vee \neg\phi_k \vee \phi_{k+j}$, for some $j: 1 \le j \le p$. □

A simpler formulation of Theorem 2 is given by:

<u>THEOREM 2´</u>. The class $\mathbb{K}$ of all ^-subalgebras of algebras belonging to a quasivariety $\mathbb{Q}$ is a quasivariety.

The following problems arise about the class $\mathbb{K}$:

1) give a convenient description of the quasivariety $\mathbb{K}$;
2) under what conditions $\mathbb{K}$ is a variety, when $\mathbb{Q}$ is a variety

<u>Remark.</u> If the languages L_1 and L_2 contain relational symbols, then Theorem 1 still holds. In this case, for a relational symbols p, we put p^ to be an L_2-formula with no more than n free variables ($|p| = n$). Then, in addition of (1.1), we put

$$P_{\mathcal{A}}(a_1,\dots,a_n) = T \text{ iff } P^{\wedge}_{\mathcal{B}}(a_1,\dots,a_n) = T.$$

Furthermore, if $p^{\wedge}$ is an atomic formula, then Theorem 2 holds too (with corresponding preformulations, of course).

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