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ON A CLASS OF NORMAL SEMIGROUPS

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A semigroup S is called normal if $xS = Sx$ for all elements x of S (S.Schwarz [1]). It is considered in this note the class of normal semigroups with the property $Sx = Sx^2$ for all x of S. Two characterizations for the semigroups of this class are obtained here.

1°. If a semigroup S is normal and regular, then $Sx = Sx^2$ for all x of S.

Proof. If $x \in S$ and $x = xyx$, then

$$Sx = Sxyx \subseteq SxSx = S^2x^2 \subseteq Sx^2 = Sx.$$

The following example shows that the converse does not hold: If S is a semigroup such that $|S| > 1$ and $|S^2| = 1$, then obviously $Sx = xS = Sx^2$ and S is not regular.

2°. A semigroup S is normal and has the property $Sx = Sx^2$ for all $x \in S$ if and only if S is an inflation of a semilattice of groups.

Proof. Let S be a normal semigroup with the property $Sx = Sx^2$ for all $x \in S$. Denote by T the set of all regular elements of S. We shall prove that $T = S^2$. If $z \in S^2$, then there exist $x, y, u, v, s, t \in S$ such that

$$z = xy = uy^2 = uy^2v = xyv = (xy)^2s = xytxy,$$

which means that $z \in T$, i.e. $S^2 \subseteq T$. The inclusion $T \subseteq S^2$ is obvious. Now we shall prove that T is normal If $x,y \in T$, then there exist $s,u \in S$ such that $xy = sx = sxusx \in S^2x = Tx$ and this implies that $xT \subseteq Tx$. By symmetry $Tx \subseteq xT$ and thus the semigroup $S^2 = T$ is regular and normal. According to [4], S^2 is a semilattice of groups. We note that idempotents of S are in the centre of S [1], and thus the set of idempotents E is a subsemigroup of $S^2 = T$.

Define a transformation ϕ of S as follows. If $x \in S$, then $x^2 \in S^2$ and thus there exists an idempoten e such that $x^2 \in G_e$. Then $\phi(x) = xe$ defines a transformation of S.

If $x,y \in S$ and $x^2 \in G_e$, $y^2 \in G_f$, then:

$$(xy)^2ef = xyxyef = xyxeyf = (xy)^2$$

and thus $(xy)^2 \in G_{ef}$, i.e.

$$\phi(xy) = xyef = xe \cdot yf = \phi(x)\phi(y).$$

Moreover, there exist $u,v \in S$ such that

$$\phi(xy) = xyef = x(ye)f = x(uy^2)f =$$
$$= xuy^2 = xye = vx^2e = vx^2 = xy.$$

Therefore ϕ is an endomorphism of S which fixes the elements of S^2 and this implies that S is an inflation of S^2.

Conversely, assume that T is a semilattice of groups, and S is an inflation of T. Then, clearly, T is a normal semigroup such that $Tt = Tt^2$ for each $t \in T$, and this implies that S is also a normal semigroup satisfying the equality $Sx = Sx^2$ for every x of S.

3°. Let S be a normal semigroup. The following statements are equivalent:

(i) $Sx = Sx^2$ for all $x \in S$;

(ii) $N(x) = \{y \in S \mid Sx \subseteq Sy\}$ for all $x \in S$.

Proof. (i)$\Longrightarrow$(ii). First we shall prove that

$$F = \{y \in S \mid Sx \subseteq Sy\}$$

is a filter which contains the element x.

Let $y, z \in F$. Then $Sx \subseteq Sy$ and $Sx \subseteq Sz$ and since

$$Sx = Sx^2 = Sx^4 = Sxx^2x \subseteq SxSx \subseteq SySz = S^2yz \subseteq Syz,$$

it follows that $yz \in F$.

Conversely, if $yz \in F$, then $Sx \subseteq Syz \subseteq Sz$ and $Sx \subseteq Syz = ySz \subseteq yS = Sy$ which means that $y, z \in F$, i.e. F is a filter. Since $Sx \subseteq Sx$, it follows that $x \in F$ and this implies that $N(x) \subseteq F$. To show the inclusion $F \subseteq N(x)$ we use II.2.10 of [3]. If $y \in F$, then $x^2 \in Sx \subseteq Sy \subseteq J(y)$. Since $x^2 \in N_1(x) \cap J(y)$, we get $y \in N_2(x) \subseteq N(x)$ and so $F \subseteq N(x)$. Hence $F = N(x)$.

(ii)$\Longrightarrow$(i) Obviosly $Sx^2 \subseteq Sx$ for any $x \in S$. Since $x^2 \in N(x)$, it follows that $Sx \subseteq Sx^2$. Therefore $Sx = Sx^2$.

R E F E R E N C E S

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ЗА ЕДНА КЛАСА НОРМАЛНИ ПОЛУГРУПИ

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Во оваа статија се разгледани нормалните полугрупи ($xS=Sx$ за секој $x\in S$) кои го задоволуваат условот $Sx=Sx^2$.

Ако полугрупата S е нормална и регуларна, тогаш $Sx=xS=Sx^2$. Меѓутоа обратното не важи. На пример, ако S е полугрупа таква што $|S|>1$ и $|S^2|=1$, тогаш $Sx=xS=Sx^2$, но S не е регуларна (1^o), со што покажуваме дека оваа класа е поширока отколку кога S е полумрежа од групи. За класата полугрупи $Sx=xS=Sx^2$ добиваме две карактеристики:

S е нормална полугрупа со својството $Sx=Sx^2$ ако и само ако S е инфлација на полумрежа од групи (2^o).

Ако S е нормална, тогаш следниве искази се еквивалентни:

(i) за секој $x\in S$, $Sx=Sx^2$

(ii) за секој $x\in S$, $N(x)=\{y\in S \mid Sx\subseteq Sy\}$ (3^o).