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ON SEPERATIVE — n-SEMIGROUPS

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The purpose of this paper is to show that the wellknow characteristics of semilattices of cancellative semigroups could be generalized on the class of n-semigroups for $n > 2$.

1. Previous definitions and the formulation of the main result.

An algebra S with an asocijative n-ary operation

$$(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$$

is said to be an n-semigroup.

A semigroup S is cancellative if for any $x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n, b \in S$

$$x_1 x_2 \ldots x_{i-1} a x_{i+1} \ldots x_n = x_1 \ldots x_{i-1} b x_{i+1} \ldots x_n \Rightarrow x = y$$

for all $i = 1, 2, \ldots, n$.

A semigroup S is separative if for any $x, y \in S$

$$x^n = x^{i-1} y x^{n-i}, \; y^n = y^{i-1} x y^{n-i} \Rightarrow x = y$$

for all $i = 1, 2, \ldots, n$.

The main result in this paper is the following theorem.

An n-semigroup S is separative if and only if S is a semilattice of cancellative n-semigroups.

3. Some results on separative n-semigroups

We shall proceede by proving some results which are usuful for proving the main result.

2. 1. If S is a separative n-semigroup, than for any $u_1, u_2, \ldots, \ldots, u_{n-1}, a, b \in S$ holds:

$$u_{k+1} \ldots u_{n-1}\, au_1 \ldots u_k = u_{k+1} \ldots u_{n-1}\, bu_1 \ldots u_k \Leftrightarrow$$
$$\Leftrightarrow u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1} = u_k \ldots u_{n-1}\, bu_1 \ldots u_{k-1}.$$

proof; If the identity $u_{k+1} \ldots u_{n-1}\, au_1\, u_2 \ldots u_k = u_{k+1} \ldots u_{n-1}\, bu_1 \ldots u_k$ is multilplied from the left by

$$(u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1})^{i-2}\, u_k \ldots u_{n-1}\, au_1 \ldots u_k$$

and from the right by

$$u_{k+1} \ldots u_{n-1}\, au_1 \ldots u_{k-1}\, (u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1})^{n-i-1}$$

we obtain

$$(u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1})^n = (u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1})^{i-1}\, (u_k \ldots$$
$$\ldots u_{n-1}\, \mathrm{b} \ldots u_{k-1})\, (u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1})^{n-i}$$

for every $i = 1, 2, \ldots, n$.

Similarly we obtain the following equality

$$(u_k \ldots u_{n-1}\, bu_1 \ldots u_{k-1})^n = (u_k \ldots u_{n-1}\, \mathrm{b}u_1 \ldots u_{k-1})^{i-1}\, (u_k \ldots u_{n-1}\, au_1 \ldots$$
$$\ldots u_{k-1})\, (u_k \ldots u_{n-1}\, bu_1 \ldots u_n)^{n-i}$$

for every $i = 1, 2, \ldots, n$.

From the last two identities, as a consepuence of the separativity of S we obtain:

$$u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1} = u_k \ldots u_{n-1}\, bu_1 \ldots u_{k-1}$$

Similarly it could be proven that the identity

$$u_k \ldots u_{n-1}\, au_1 \ldots u_{k-1} = u_k \ldots u_{n-1}\, bu_1 \ldots u_{k-1}$$

implies the identity

$$u_{k+1} \ldots u_{n-1}\, au_1 \ldots u_k = u_{k+1} \ldots u_{n-1}\, bu_1 \ldots u_{k1}.$$

2.2. If S is a separative n-semigroup, then for any $u_1, u_2, \ldots, u_{n-1}, x_1, x_2, \ldots, x_n, a, b, x, \in S$ the following conditions hold:

(i) $u_1 u_2 \dots u_{i-1} x^n u_i \dots u_{n-2} a = u_1 u_2 \dots u_{i-1} x^n u_i \dots$

$\dots u_{n-2} b \Leftrightarrow u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} a = u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} b$

(ii) $u_1 u_2 \dots u_{i-1} x_1 x_2 \dots x_n u_i \dots u_{n-2} a = u_1 u_2 \dots u_{i-1} x_1 x_2 \dots$

$\dots x_n u_i \dots u_{n-2} b \Leftrightarrow u_1 u_2 \dots u_{i-1} x_2 \dots x_n x_1 u_1 \dots u_{n-2} a =$

$= u_1 u_2 \dots u_{i-1} x_2 \dots x_n x_1 u_i \dots u_{n-2} b$

(iii) $u_1 u_2 \dots u_{i-1} x_1 x_2 \dots x_n u_i \dots u_{n-2} a = u_1 u_2 \dots$

$\dots u_{i-1} x_1 x_2 \dots x_n u_i \dots u_{n-2} b \Leftrightarrow u_1 u_2 \dots u_{i-1} x_{i_1} x_{i_2} \dots x_{i_n} u_i \dots$

$\dots u_{n-2} a = u_1 u_2 \dots u_{i-1} x_{i_1} x_{i_2} \dots x_{i_n} u_i \dots u_{n-2} b$

where $(i_i, i_2, \dots, i_n)$ is some permutation of numbres $1, 2, \dots, n$.

(iV) $u_1 u_2 \dots u_{i-1} x_1^{j_1} x_1^{j_2} \dots x_k^{j_k} u_i \dots u_{n-2} a = u_1 u_2 \dots$

$\dots u_{i-1} x_1^{j_1} x_1^{j_2} \dots x_k^{j_k} u_i \dots u_{n-2} b \Leftrightarrow u_1 u_2 \dots u_{i-1} x_1^{i_1} x_2^{i_2} \dots$

$\dots x_k^{i_k} u_i \dots u_{n-2} a = u_1 u_2 \dots u_{i-1} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} u_i \dots u_{n-2} b$

where $i_1 + i_2 + \dots + i_k = j_1 + j_2 + \dots + j_k = n$.

proof: Let $u_1 u_2 \dots u_{i-1} x^n u_i \dots u_{n-2} a = u_1 \dots u_{i-1} x^n u_i \dots u_{n-2} b$.

According to **2.1.** we have

$x u_i \dots u_{n-2} a u_1 u_2 \dots u_{i-1} x^{n-1} = x u_i \dots u_{n-2} b u_1 u_2 \dots u_{i-1} x^{n-1}$

from what follows

$(u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} a) u_1 u_2 \dots u_{i-1} x^{n-1} u_i \dots u_{n-2} a =$

$= (u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} b) u_1 u_2 \dots u_{i-1} x^{n-1} u_i \dots u_{n-2} b$

Denote bu α the element $u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} a$ and β the element $u_1 u_2 \dots u_{i-1} x u_i \dots u_{n-2} b$, then the previous identity takes the following form:

$\alpha u_1 u_2 \dots u_{i-1} x^{n-1} u_i \dots u_{n-2} a = \beta u_1 u_2 \dots u_{i-1} x^{n-1} u_i \dots u_{n-2} a$

Appllying **2.1.** from the last identity we obtain

$\alpha^2 u_1 u_2 \dots u_{i-1} x^{n-2} u_i \dots u_{n-2} a = \alpha\beta u_1 u_2 \dots u_{i-1} x^{n-2} u_i \dots u_{n-2} a$

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Applying the same procedure (n-2) times we obtain:

$$\alpha^n = \alpha^{n-1}\ \beta$$

By analogy we obtain the following result:

$$\beta^n = \beta^{n-1}\alpha$$

From the last two identities, as a consequence of the separativity of S we obtain: $\alpha = \beta$ *i.e.*

$$u_1\ u_2 \dots u_{i-1}\ xu_i \dots u_{n-2}\ a = u_1\ u_2 \dots u_{i-1}\ xu_i \dots u_{n-2}\ b.$$

Conversely, assume that the last identity holds. If we aply **2.1**, we obtain

$$u_i \dots u_{n-2}\ au_1\ u_2 \dots u_{i-1}\ x = u_i \dots u_{n-2}\ bu_1\ u_2 \dots u_{i-1}\ x$$

from what follows

$$u_1\ u_2 \dots u_{i-1}\ x^n\ u_i \dots u_{n-2}\ a = u_1\ u_2 \dots u_{i-1}\ x^n\ u_i \dots u_{n-2}\ \mathrm{b}.$$

(ii) Let $u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_n\ u_k \dots u_{n-2}\ a =$

$$= u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_n\ u_k \dots u_{n-2}\ b$$

According to **2.1.** this identity is equivalent to

$$u_k \dots u_{n-2}\ au_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_n = u_k \dots u_{n-2}\ \mathrm{b}u_1\ u_2 \dots$$
$$\dots u_{k-1}\ x_1\ x_2 \dots x_n$$

If we multiply by $x_1\ x_2 \dots x_{n-1}$ from the left and apply **2.1**., and then mltiply by $u_k \dots u_{n-2}\ au_1\ u_2 \dots u_{k-1}$ from the left we obtain

$$u_k \dots u_{n-2}\ a\ (u_1\ u_2 \dots u_{k-1}\ x_n\ x_1\ x_2 \dots x_{n-1}\ u_k \dots u_{n-2}\ a)\ u_1\ u_2 \dots$$
$$\dots u_{k-1}\ x_1\ x_2 \dots x_{n-1} = u_k \dots u_{n-2}\ a\ (u_1\ u_2, \dots u_{k-1}\ x_n\ x_1\ x_2 \dots$$
$$\dots x_{n-2}\ b)\ u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_{n-1}$$

Denoting the element $u_1\ u_2 \dots u_{k-1}\ x_n\ x_1\ x_2 \dots x_{n-1}\ u_k \dots u_{n-2}\ a$ by α and $u_1\ u_2 \dots u_{k-1}\ x_n\ x_1\ x_2 \dots x_{n-1}\ u_k \dots u_{n-2}\ b$ by β, the last identity takes the following form:

$$u_k \dots u_{n-2}\ a\,\alpha\, u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_{n-1} = u_k \dots u_{n-2}\ a\,\beta\, u_1\ u_2 \dots$$
$$\dots u_{k-1}\ x_1\ x_1 \dots x_{n-1}$$

If we apply **2,1.** On the last equality, then mitiplying by $x_2 \ldots x_n$ from the left, and then again apply **1.2,** we obtain:

$$(x_n\ x_1\ x_2 \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a)\ \alpha\, u_1\ u_2 \ldots u_{k-1}\ x_2 \ldots x_{n-1} =$$
$$= (x_n\ x_1\ x_2 \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a)\ \beta\, u_1\ u_2 \ldots u_{k-1}\ x_2 \ldots x_{n-1}$$

we procede by mltiplying by $u_k \ldots u_{n-2}\ au_1\ u_2 \ldots u_{k-1}$ from the left side and then, applying **2.1,** we obtain

$$x_2\ x_3 \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a\,\alpha^2\ u_1\ u_2 \ldots u_{k-1} = x_2\ x_3 \ldots x_{n-1}\ u_k \ldots$$
$$\ldots u_{n-2}\ a\,\alpha\ \beta\, u_1\ u_2\ \ldots u_{k-1}$$

By induction we prove that our previous result holds. Let

$$x_i\ x_{i+1} \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a\,\alpha^i\ u_1\ u_2 \ldots u_{k-1} =$$
$$= x_i\ x_{i+1} \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a\,\alpha^{i-1}\ \beta\, u_1\ u_2 \ldots u_{k-1}$$

If we multiply by $x_{i+1} \ldots x_n\ x_1\ x_2 \ldots x_{i-1}$ from the left and then apply **2.1,** and then again multply by $u_k \ldots u_{n-2}\ au_1\ u_2 \ldots u_{k-1}$ from the left, we obtain

$$u_k \ldots u_{n-2}\ a\,\alpha^{i+1}\ u_1\ u_2 \ldots u_{k-1}\ x_{i+1} \ldots x_{n-1} =$$
$$= u_k \ldots u_{n-2}\ a\,\alpha^i\ \beta u_1 \ldots u_{k-1}\ x_{i+1} \ldots x_{n-1}$$

Containing the procedure we have

$$\alpha^n = \alpha^{n-1}\ \beta$$

By simetry, analogous we obtain:

$$\beta^n = \beta^{n-1}\ \alpha$$

Since S is a seperative n-semigroup, from the last two identities we obtain

$$\alpha = \beta$$

i. e.

$$u_1\ u_2 \ldots u_{k-1}\ x_n\ x_1\ x_2 \ldots x_{n-1}\ u_k \ldots u_{n-2}\ a =$$
$$= u_1\ u_2 \ldots u_{k-1}\ x_n\ x_1\ x_2 \ldots x_{n-1}\ u_k \ldots u_{n-2}\ \mathbf{b}.$$

(iii) Let

$$u_1\ u_2 \ldots u_{k-1}\ x_1\ x_2 \ldots x_m\ x_{m+1} \ldots x_n\ u_k \ldots u_{n-2}\ a =$$
$$= u_1\ u_2 \ldots u_{k-1}\ x_1\ x_2 \ldots x_m\ x_{m+1} \ldots x_n\ u_k \ldots u_{n-2}\ b$$

According to **(i)**, this is equivalent to

$$u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots \overset{n-1}{(x_m\ x_{m+1})}\ x_{m+1} \dots x_n\ u_k \dots u_{n-2}\ a =$$

$$= u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots \overset{n-1}{(x_m\ x_{m+1})}\ x_{m+1} \dots x_n\ u_k \dots u_{n-2}\ b$$

and according to **(ii)** this is equivalent to

$$u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots \overset{n-1}{(x_{m+1}\ x_m)}\ x_{m+1} \dots x_n\ u_k \dots u_{n-2}\ a =$$

$$= u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots \overset{n-1}{(x_{m+1}\ x_m)}\ x_{m+1} \dots x_n\ u_k \dots u_{n-2}\ b$$

If we again apply **(ii)** on the last identity we obtain

$$u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_{m+1}\ x_m \dots x_n\ u_k \dots u_{n-2}\ a =$$

$$= u_1\ u_2 \dots u_{k-1}\ x_1\ x_2 \dots x_{m+1}\ x_m \dots x_n\ u_k \dots u_{n-2}\ b$$

The fact that every permutation is product of transpositions, we obtain that **(iii)** is corect.

(iv) Let

$$u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_k}\ u_k \dots u_{n-2}\ a = u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots$$
$$\dots x_k^{i_k}\ u_k \dots u_{n-2}\ b$$

If we apply **(i)** then we obtain

$$x_k\ u_k \dots u_{n-2}\ a u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_k-1} =$$

$$= x_k\ u_k \dots u_{n-2}\ b u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_k-1}$$

Multiplying the last identity by

$$u_k \dots u_{n-2}\ a u_1\ u_2 \dots u_{k-1}\ x_1^{j_1}\ x_2^{j_2} \dots x_k^{j_k-1}$$

and denoting by σ the element

$$u_1\ u_2 \dots u_{k-1}\ x_1^{j_1}\ x_2^{j_2} \dots x_k^{j_k}\ u_k \dots u_{n-2}\ a,$$

and by β the element

$$u_1\ u_2 \dots u_{k-1}\ x_1^{j_1}\ x_2^{j_2} \dots x_k^{j_k}\ u_k \dots u_{n-2}\ b$$

we obtain

$$u_k \dots u_{n-2}\ a\,\alpha\, u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_{k-1}} =$$

$$= u_k \dots u_{n-2}\ a\beta u_1\ u_2 \dots u_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_{k-1}}$$

Following the same procedure we obtain

$$\alpha^{i_1 + i_2 + \dots + i_k} = \alpha^{i_1 + i_1 + \dots + i_{k-1}}\beta$$

i. e.

$$\alpha^n = \alpha^{n-1}\ \beta$$

The same procedure yields the folowing result:

$$\beta^n = \beta^{n-1}\ \alpha$$

Since, S is separative, the last two identities implies:

$$u_1\ u_2 \dots u_{k-1}\ x_1^{j_1}\ x_2^{j_2} \dots x_k^{j_k}\, u_k \dots u_{n-2}\ a$$

$$= u_1\ u_2 \dots u_{k-1}\ x_1^{j_1}\ x_2^{j_2} \dots x_k^{j_k}\ u_k \dots u_{n-2}\ b$$

Sinilarly we can prove, that the last identity implies the following identity:

$$u_1\ u_2 \dots x_{k-1}\ x_1^{i_1}\ x_2^{i_2} \dots x_k^{i_k}\, u_k \dots u_{n-2}\ a$$

$$= u_1\, u_2 \dots u_{k-1}\, x_i^{i_1}\ x_i^{i_2} \dots x_k^{i_k}\ u_k \dots u_{n-2}\ b$$

3, The proof of the main result

Assuming that S is a separative n-semigroup, we shall proof that S is an n-semilattice of cancellative n-semigroup.

Define a relation σ in S by.

$$x\sigma\ y \text{ if and only if for all } u_1,\ u_2, \dots u_{n-2},\ a,\ b \in S$$

$$u_1\ u_2 \dots u_{k-1}\ xu_k \dots u_{n-2}\ a = u_1\ u_2 \dots u_{k-1}\ xu_k \dots u_{n-2}\ b \Leftrightarrow$$

$$\Leftrightarrow u_1\ u_2 \dots u_{k-1}\ yu_k \dots u_{n-2}\ a = u_1\ u_2 \dots u_{k-1}\ yu_k \dots u_{n-2}\ b \qquad (1)$$

It is clear that σ is an equivalnce relation. We have to prove that σ is also a congruence.

Let $x \sigma y$ and $z_1, z_2, \ldots, z_{n-1}$, be arbitrary elements of S and let

$$u_1 u_2 \ldots u_{k-1} (z_1 z_2 \ldots z_{i-1} y z_i \ldots z_{n-1}) u_k \ldots u_{n-2} a=$$

$$= u_1 u_2 \ldots u_{k-1} (z_1 z_2 \ldots z_{i-1} x z_i \ldots z_{n-1}) u_k \ldots u_{n-2} b$$

Using **2.2.** (ii) and definition of σ, we obtain that the last identity is equivalent to:

$$u_1 u_2 \ldots u_{k-1} y (z_i \ldots z_{n-1} z_1 z_2 \ldots z_{i-1} u_k) u_{k+1} \ldots u_{n-2} a=$$

$$= u_1 u_2 \ldots u_{k-1} x (z_i \ldots z_{n-1} z_1 z_2 \ldots z_{i-} u_k) u_{k+1} \ldots u_{n-2} b$$

Applying **2.2.** (ii) again we obtain

$$u_1 u_2 \ldots u_{k-1} (z_1 z_2 \ldots z_{i-1} y z_i \ldots z_{n-1}) u_k \ldots u_{n-2} a=$$

$$= u_1 u_1 \ldots u_{k-1} (z_1 z_2 \ldots z_{i-1} x z_i \ldots z_{n-1}) u_k \ldots u_{n-2} b$$

this concludes our proof that σ is a congruence.
From **(i)** it follows that $x^n \sigma x$ for all $x \in S$ which proves that S/σ is a band.

From **(ii)** follows that $x_1 x_2 \ldots x_n \sigma x_{i_1} x_{i_2} \ldots x_{i_n}$ where

$(i_1, i_2, \ldots, i_n)$ is some permutation of $1, 2, \ldots, n$.

From **(iv)** it followos $x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} \sigma x_1^{j_1} x_1^{j_2} \ldots x_k^{j_k}$ where $i_1+i_2+\ldots+$

$+i_k=j_1+j_2+\ldots+j_k=n$

Thus we have proven that S/σ is a semelattice.

It remains to proof that every class is cancellative.

Let $u_1 u_2 \ldots u_{k-1} x u_k \ldots, u_{n-1} = u_1 u_2 \ldots u_{k-1} y u_k \ldots u_{n-1}$

and let $u_1, u_2, \ldots, u_{n-1}, x, y$ belong to same class.

From $u_1 \sigma u_2 \sigma u_3 \ldots \sigma u_{n-1} \sigma x$ we obtain

$$x^n = x^{k-1} y x^{n-k}$$

and from $u_1 \sigma u_2 \sigma u_3 \ldots \sigma u_{n-1} \sigma y$ we obtain

$$y^n = y^{k-1} x y^{n-k}$$

Since S is separative, from the last two identities follows that $x=y$. This canceludes our proof that every separative n-semigroup is an n-semilettice of cancellative n-semigroups. Conversely, assume that S is a semilattice of cancellative n-semigroups. We shall proof that S is a separative.

Let τ be an n-semilattice congruence, defined on S, which all classes are cancellative. Let $x, y \in S$,

$$If\ x^n = x^{i-1}\ yx^{n-i} \text{ and } y^n = y^{i-1}\ xy^{n-i}.$$

Since τ is an n-semilattice congruence we have

$$x\tau x^n = x^{i-1}\ yx^{n-i}\ \tau yx^{n-1}\ \tau y^{n-1}x = y^n\ \tau y$$

from what follows that

$$x,\ x^n,\ x^{i-1}\ yx^{n-i},\ y^{i-1}\ xy^{n-i},\ y^n,\ y$$

are in the same class S_τ, so they could be cancelled, and S is separative.

Note. The congruence defined with (1) is the greatest n-semilattice congruence, which classes are cancelative.

Let ζ be an arbitrary semilattice congruence which classes are cancellative. Let

$$x \xi y \text{ and } u_1\ u_2 \ldots u_{k-1}\ y\ u_k \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ b$$

Since $\xi$is a congruence we obtain

$$u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a\ \xi\ u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ a =$$
$$= u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ b\ \xi\ u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ b$$

Thus the elements

$$u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a,\ u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ a,$$
$$u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ b$$

belong to the same class.

Assume that $u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots$

$$\ldots u_{n-2}\ \mathrm{b}$$

If we multiply by $u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}$ from the right and apply **(ii)** from **2.2-** we obtain

$$u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ (u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ a) =$$
$$= u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ (u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ \mathrm{b})$$

Denote

$$\alpha = u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ a$$
$$\beta = u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ \mathrm{b}$$

The last identity could be written in the following form:

$$u_1\ u_1 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ \alpha = u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ \beta$$

If we apply **2.1.** On the previous identity, and then multiply by $a\ (u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a)^{n-2}$ from the left we obain

$\alpha\ (u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a)^{n-1} = \beta\ (u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a)^{n-1}$

Hence the cancellation law in the ξ-class containing

$$u_1\ u_2 \ldots u_{k-1}\ x_k \ldots u_{n-2}\ a,\ u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-1}\ a,$$

$u_1\ u_1 \ldots u_{k-1}\ xu_k \ldots u_{n-2}$ b implies

$$u_1\ u_2 \ldots u_{k-1}\ yu_{k-2}\ a = u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ b.$$

By simetry, we conclude that the identity

$$u_1\ u_2 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ a = u_1\ u_1 \ldots u_{k-1}\ yu_k \ldots u_{n-2}\ b$$

implies

$$u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{k-1}\ xu_k \ldots u_{n-2}\ b$$

which shows that $x\sigma y$. Therfore $\xi \leqslant \sigma$.

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ЗА СЕПАРАТИВНИТЕ *n*-ПОЛУГРУПИ

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Резиме

Во оваа работа ги рагледуваме сепаративните *n*-полугрупи кои се обопштени од класата сепаративни полугрупи,

За една *n*-полтгртпа *S* велиме дека е кратлива ако и само ако во *S* е исполнет следниов квазиидентитет:

$$a_1\ a_2 \ldots a_{i-1}\ xa_{i+1} \ldots a_n = a_1\ a_2 \ldots a_{i-1}\ ya_{i+1} \ldots a_n \Rightarrow x = y.$$

Една n-полугррпа се вика сепаративна ако за кои било $x, y \in S$ важи;

$$x^n = x^{i-1}\ y x^{n-i} \text{ и } y^n = y^{i-1}\ x y^{n-i} \Rightarrow x = y.$$

Ако S е сепаративна n-полугрупа, тогаш за кои било $u_1, u_2, \ldots, \ldots, u_{n-1}\ a, b, x_1, x_2, \ldots, x_n \in S$ Точни се следните искази;

(i) $u_{i+1} \ldots u_{n-1}\ a u_1 \ldots u_i = u_{i+1} \ldots u_{n-1}\ b u_1\ u_2 \ldots u_i \Leftrightarrow$

$\Leftrightarrow u_i \ldots u_{n-1}\ a u_1 \ldots u_{i-1} = u_i \ldots u_{n-1}\ b u_1\ u_2 \ldots u_{i-1}$

(ii) $u_1\ u_2 \ldots u_{i-1}\ x^n\ u_i \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ x^n\ u_i \ldots u_{n-2}\ b \Leftrightarrow$

$\Leftrightarrow u_1\ u_2 \ldots u_{i-1}\ x u_i \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ x u_i \ldots u_{n-2}\ b$

(iii) $u_1\ u_2 \ldots u_{i-1}\ x_1\ x_2 \ldots x_n\ u_i \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ x_1\ x_2 \ldots$

$\ldots x_n\ u_i \ldots u_{n-2}\, b \Leftrightarrow u_1\ u_2 \ldots u_{i-1}\ x_2 \ldots x_n\ x_1\ u_i \ldots u_{n-2}\ a =$

$= u_1\ u_2 \ldots u_{i-1}\ x_2 \ldots x_n\ x_1\ u_i \ldots u_n\ b$

(iv) $u_1\ u_2 \ldots u_{i-1}\ x_1\ x_2 \ldots x_n\ u_i \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ x_1\ x_2 \ldots$

$\ldots x_n\ u_i \ldots u_{n-2}\ b \Leftrightarrow u_1\ u_2 \ldots u_{i-1}\ x_{k_1}\ x_{k_2} \ldots x_{k_n}\ u_i \ldots$

$\ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ x_{k_1} \ldots x_{k_n}\ u_i \ldots u_{n-2}\ b$

каде што $(k_1, k_2, \ldots, k_n)$ е некоја пермутација од броевите $1, 2, \ldots n$

(v) $u_1\ u_2 \ldots u_{i-1}\ \overset{j_1}{x_1}\ \overset{j_2}{x_2}\ {}_2 \ldots \overset{j_k}{x_k}\ u_i \ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ \overset{j_1}{x_1}\ \overset{i_2}{x_2} \ldots$

$\ldots \overset{j_k}{x_k}\ ui \ldots u_{n-2}\ b \Leftrightarrow u_1\ u_2 \ldots u_{i-1}\ \overset{i_1}{x_1}\ \overset{i_2}{x_2} \ldots \overset{i_k}{x_k}\ u_i \ldots$

$\ldots u_{n-2}\ a = u_1\ u_2 \ldots u_{i-1}\ \overset{i_1}{x_1}\ \overset{i_1}{x_1} \ldots \overset{i_k}{x_k}\ u_i \ldots u_{n-2}\ b$

каде што $i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k = n$

Дефинираме релација σ во S со:

$x \sigma y$ за секои $u_1, u_2, \ldots, u_{n-2}, a, b \in S$ (1)

$$u_1\, u_2 \ldots u_{i-1}\, xu_i \ldots u_{n-2}\, a = u_1\, u_2 \ldots u_{i-1}\, xu_i \ldots u_{n-2}\, b \Leftrightarrow$$

$$\Leftrightarrow u_1\, u_2 \ldots u_{i-1}\, yu_i \ldots u_{n-2}\, a = u_1\, u_2 \ldots u_{i-1}\, yu_i \ldots u_{n-2}\, b$$

тогаш S/σ е n-полумрежа од n-полугрупи т. е. точна е следната теорема:

S е сепаративна ако и само ако S е n-полумрежа од кратливи n-полугрупи.

Релацијата дефинирана со (1) е најголемата n-полумрежна конгруенција.