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n-SUBSEMIGROUS OF SEMIGROUPS SATISFYING THE IDENTITY $x^r = x^{r+m}$

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A subset Q of a semigroup S is said to be an n-*subsemigroup* of S if $Q^{n+1} \subseteq Q$. If **C** is a class of semigroups, then by **C** (n) is denoted the class of n-subsemigroups of **C**-semigroups. Let $\mathbf{P}_{r,m}$ be the variety of semigroups defined in the title, and $\mathbf{C}_{r,m}$ be the variety of commutative $\mathbf{P}_{r,m}$-semigroups. Main results :$\mathbf{P}_{r,sn}$ (n), $\mathbf{P}_{o,m}$ (n), $\mathbf{P}_{1,m}$ (n) and $\mathbf{C}_{r,m}$ (n) are varieties for any r, s, m, n; if $r \neq 0,1$ and n is not a divisor of m, then $\mathbf{P}_{r,m}$ (n) is not a variety.

O. Preliminary definitions and main results

O. 1. An algebra Q [. . .] with an n + 1-ary operation

$$[\ldots] : (x_0, \ldots, x_n) \mapsto [x_0 \ldots x_n]$$

is called an n-*semigroup* if the operation is associative, i. e. if the following identity equations are satisfied:

$$[[x_0 \ldots x_n]\, x_{n+1} \ldots x_{2n}] = [x_0 [x_1 \ldots x_{n+1}]\, x_{n+2} \ldots x_{2n}] = \ldots$$
$$= [x_0 \ldots x_{n-1} [x_n \ldots x_{2n}]].$$

Then, all the continued products on a sequence $a_0, \ldots, a_{sn}$ of elements of Q are equal, and the result of such a product is denoted by $[a_0 \ldots a_{sn}]$; if $s = 0$, then $[a_0] = a_0$. An n-semigroup is said to be *commutative* if for any permutation $i_0, \ldots, i_n$ of 0, 1,..., n the following identity is satisfied:

$$[x_0 \ldots x_n] = [x_{i_0} \ldots x_{i_n}].$$

If $s \geqslant 0$ and if $j_0 \ldots j_{sn}$ is a permutation of 0, 1,..., sn, then the following identity satisfies every commutative n-semigroup:

$$[x_0 \ldots x_{sn}] = [x_{j_0} \ldots x_{j_{sn}}].$$

Throughout the paper we will usually write „n-semigroup Q“ instead of „n-semigroup Q [. . .]“.

0.2. Let S be a semigroup and Q a subset of S such that $Q^{n+1} \subseteq Q$. Then Q is called an *n-subsemigroup of S*. Clearly, the semigroup operation induces on Q an associative $n+1$-ary operation, and the corresponding n-semigroup Q is said to be *induced* by the given semigroup S. If, in addition, Q is a generating subset of S, then S is caled a *covering semigroup* of Q; and this covering is *proper* if $Q^i \cap Q^j \neq \emptyset \Rightarrow i = j \pmod n$. We note that the class of proper coverings of an n-semigroup is not empty and that a commutative n-semigroup admits proper commutative coverings. (These results, and convenient descriptions of some other classes of n-semigroups, can be found in [2] and [3].)

If $\mathbf{C}$ is a class of semigroups, then by $\mathbf{C}(n)$ is denoted the class of n-semigroups which can be embedded in $\mathbf{C}$-semigroups as n-subsemigroups. For example, if $\mathbf{C}$ is the variety of (commutative) semigroups then $\mathbf{C}(n)$ is the variety of (commutative) n-semigroups.

0. 3. Let $n \geqslant 1$, $r \geqslant 0$ and $m \geqslant 1$ be given integers. Denote by $\mathbf{P}_{r,m}$ the variety of semigroups which satisfy the identity equation $x^r = x^{r+m}$, and by $\mathbf{C}_{r,m}$ the variety of commutative $\mathbf{P}_{r,m}$-semigroups. ($\mathbf{P}_{o,m}$ is the variety of semigroups which satisfy the identities $x^m y = y$, $y x^m = y$, and it is, in fact, the class of groups in which the orders of all elements are divisors of m.)

The following theorems are the main results of this paper.

Teorem 1. $\mathbf{P}_{r,m}$ *(n) is a variety iff n is a divisor of m or* $r \in \{0,1\}$.

Theorem 2. $\mathbf{C}_{r,m}$ *(n) is a variety.*

The proofs of these results are given in the sections **1-4.** In the section **5** it is shown that each of the varieties $\mathbf{P}_{r,sn}(n)$, $\mathbf{P}_{1,m}(n)$, $\mathbf{P}_{o,m}(n)$ and $\mathbf{C}_{r,m}(n)$ can be defined by a finite system of identities.

0. 4 If $\mathbf{C}$ is a variety (or more generally a quasivariety) of semigroups, then $\mathbf{C}(n)$ is a quasivariety of n-semigroups. This result is a special case of the corresponding result on quasivarieties of universal algebras (for example, [1] p. 274). We find it interesting to look for a convenient description of the set $\mathcal{V}_n$ of the varieties $\mathbf{C}$ of semigroups such that the corresponding classes $\mathbf{C}(n)$ are varieties of n-semigroups. Each of theorems **1** and **2** implies that the intersection $\mathcal{V}$ of all the sets $\mathcal{V}_n$ is an infinite set. Theorem **1** implies that the complement $\mathcal{V}'_n$ (in the set of varieties of semigroups) of $\mathcal{V}_n$ is an infinite set, for each $n \geqslant 2$.

We will state here the main results of the papers [4] and [5]. Let $\mathbf{L}_k$ ($\mathbf{R}_k$) be the variety of semigroups S such that each element of S^k is a left (right) zero in S, and let $\mathbf{O}_k = \mathbf{L}_k \cap \mathbf{R}_k$. Then $\mathbf{L}_k, \mathbf{R}_k, \mathbf{O}_k \in \mathcal{V}$, for every $k \geqslant 1$. If $\mathbf{D}^e$ ($\mathbf{D}^r$) is the variety of left (right) distributive semigroups, and $\mathbf{D} = \mathbf{D}^e \cap \mathbf{D}^r$, then $\mathbf{D}^e, \mathbf{D}^r \in \mathcal{V}'_n$ and $\mathbf{D} \in \mathcal{V}$, for every $n \geqslant 2$.

1. Here will be asumed that n is a divisor of m. As corollaries of the main result of the paper [3] we obtain the following descriptions of the classes $\mathbf{P}_{r,sn}(n)$, $\mathbf{C}_{r,sn}(n)$.

1.1. *$Q \in \mathbf{P}_{r,sn}(n)$ iff the following identity is satisfied in Q:*

$$[x_1 \dots x_i \, (x_{p+1} \dots x_q)^r \, x_{i+1} \dots x_p] = [x_1 \dots x_i \, (x_{p+1} \dots x_q)^{r+sn} \dots x_p], \quad (1.1)$$

for any integers i, p, q such that $0 \leqslant i \leqslant p < q$ and $p + r(q-p) \equiv 1 \pmod{n}$.

1.2. *$Q \in \mathbf{C}_{r,sn}(n)$ iff Q is a commutative n-semigroup which satisfies all the identities (1.1.)*

Now, from **1.1.** and **1.2.** it follows that:

1.3. *$\mathbf{P}_{r,sn}(n)$ and $\mathbf{C}_{r,sn}(n)$ are varieties.*

2. Let d be the greatest common divisor of m and n, and i, j, m_1, n_1 be integers such that:

$$in = jm + d, \; n = n_1 d, \; m = m_1 d, \; i > 0, \; j \geqslant 0.$$

The following two propositions are obvious.

2.1. *Let Q be an n-semigroup and let a $d+1$-ary operation $[\dots]'$ be defined on Q by:*

$$[x_o \dots x_d]' = [x_0^{jm+1} \; x_1 \dots x_d]. \quad (2.1)$$

If $Q[\dots]' \in \mathbf{P}_{1,m}(d)$ and if the following identity is satisfied:

$$[x_0 x_1 \dots x_n]' = [x_0 \, x_1 \dots x_n], \quad (2.2)$$

then $Q \in \mathbf{P}_{1,m}$ (n).

2.2. *If $Q \in \mathbf{P}_{1,m}(n)$ and p, s, q, κ are such integers that $0 \leqslant p \leqslant d$, $0 \leqslant q \leqslant sd$, $1 \leqslant k \leqslant sd-q+1$, then the following identities are satisfied in Q:*

$$[x_0^{jm+1} x_1 \dots x_d] = [x_0 \dots x_{p-1} \;\; x_p^{jm+1} x_{p+1} \dots x_d]; \quad (2.3)$$

$$[x_0 \, x_1 \dots x_n] = [x_0^{n_1 jm+1} \, x_1 \dots x_n]; \quad (2.4)$$

$$[x_0^{jsm+1} \, x_1 \dots x_{sd}] = [x_0^{j(s+m_1 k)m+1} x_1 \dots x_{q+k-1} (x_q \dots x_{q+k-1})^m x_{q+k} \dots x_{sd}]. \quad (2.5)$$

Now, $\mathbf{P}_{1,m}(n)$ and $\mathbf{C}_{1,m}(n)$ will be described.

2.3. *$Q \in \mathbf{P}_{1,m}(n)$ iff all the identities (2.3) — (2.5) are satisfied.*

Proof. Assume the identities $(2 \cdot 3)$ — $(2 \cdot 5)$. By a finite number of applications of (2.3) we obtain that

$$[x_0^{sjm+1} x_1 \dots x_{sd}] = [x_0 \dots x_{p-1} \;\; x_p^{sjm+1} \, x_{p+1} \dots x_{sd}]$$

is an identity for any integers s, p such that $s \geqslant 0$, $0 \leqslant p \leqslant sd$. If the operation $[\ldots]'$ is defined by (2.1), then it can be easily seen that

$$[x_0 \ldots x_{p-1} [x_p \ldots x_{p+d}]' \; x_{p+d+1} \ldots x_{2d}]' = [x_0^{2jm+1} x_1 \ldots x_{2d}],$$

and this implies that $Q[\ldots]'$ is a d-semigroup. Moreover we have:

$$[x_0 \ldots x_{sd}]' = [x_0^{sjm+1} \; x_1 \ldots x_{sd}], \tag{2.6}$$

for every $s \geqslant 0$.

Let s, q, k be such that $0 \leqslant q \leqslant sd$, $1 \leqslant sd - q + 1$. By (2.6) and (2.5) we have:

$$\begin{aligned}[x_0 \ldots x_{sd}]' &= [x_0^{sjm+1} \; x_1 \ldots x_{sd}] \\ &= [x^{j(s+m_1 k)m+1} \; x_1 \ldots x_{q+k-1} \; (x_q \ldots x_{q+k-1})^m \; x_{q+k} \ldots x_{sd}] \\ &= [x_0 \ldots x_{q-1} \; (x_q \ldots x_{q+k-1})^{m+1} \; x_{q+k} \ldots x_{sd}]',\end{aligned}$$

and this implies that $Q[\ldots]'$ satisfies (1.1), i.e. that $Q[\ldots]' \in P_{1,m}$ (d). Finaly, by **2.1** we get that $Q \in P_{1,m}$ (n).

2.4 *$Q \in \mathbf{C}_{1,m}$ (n) iff Q is a commutative n-semigroup which satisfies all the identities* (2.3)—(2.5).

Proof. The d-semigroup $Q[\ldots]'$ defined by (2.1) is also commutatioe and by 1.2 $Q[\ldots]'$ is a d-subsemigroup of a semigroup $T \in \mathbf{C}_{1,m}$. Then Q is an n-subsemigroup of T.

The following statements can be proved in the same way as the corresponding statements for the case $r = 1$.

2.1'. *Let Q be an n-semigroup, c a fixed element of Q and $[\ldots]'$ a $d+1$-ary operation on Q defined by:*

$$[x_0 \ldots x_d]' = [c^{jm} \; x_0 \ldots x_d]. \tag{2.1'}$$

If $Q[\ldots]' \in \mathbf{P}_{0,m}$ (d) and if (2.2) is satisfied then $Q \in \mathbf{P}_{0,m}$ (n).)

2.2'. *If $Q \in \mathbf{P}_{0,m}$ (n) then the following identities are satisfied*

$$[x^{jm} \; x_0 \ldots x_d] = [x_0 \ldots x_{p-1} \; y^{jm} \; x_p \ldots x_d]; \tag{2.3'}$$

$$[x_0 \ldots x_n] = [x^{jn_1 m} x_0 \ldots x_n]; \tag{2.4'}$$

$$[x^{jsm} x_0 \ldots x_{sd}] = [x^{j(s+m_1 k)m} \; x_0 \ldots x_t (x_{sd+1} \ldots x_{sd+k})^m \, x_{t+1} \ldots x_{sd}]; \tag{2.5'}$$

for any integers $p, s\, t, k$ *such that* $s \geqslant 0,\ k \geqslant 1,\ 0 \leqslant p \leqslant d+1,\ 0 \leqslant t \leqslant sd.$

2.3.'. $Q \in \mathbf{P}_{0,m}(n)$ *iff all the identities* $(2.3')-(2.5')$ *are satisfied.*

2.4'. $Q \in \mathbf{C}_{0,m}(n)$ *iff* Q *is commutative and all the identities* $(2.3')-(2.5')$ *are satisfied.*

As a summary we have the following proposition:

2.5. *The classes* $\mathbf{P}_{1,m}(n)$, $\mathbf{C}_{1\,m}(n)$, $\mathbf{P}_{0,m}(n)$ *and* $\mathbf{C}_{0,m}(n)$ *are varieties.*

3. Here we shall complete the proof of Theorem 1.

Assume that $r \neq 0,1$ and that n is not a divisor of m.

Let Σ be the set of all identities that hold in $\mathbf{P}_{r,m}$ (i.e. the identites which are consequences from the identity $x^r = x^{r+m}$), and $\Sigma(n)$ be the set of n-semigroup identities defined by:

$$\Sigma(n) = \{[x_{i_0} \dots x_{i_{pn}}] = [x_{j_0} \dots x_{j_{qn}}] \mid x_{i_0} \dots x_{i_{pn}} = x_{j_0} \dots x_{j_{qn}} \in \Sigma\}.$$

Clearly, if $Q \in \mathbf{P}_{r,m}(n)$, then Q satisfies all the identities in $\Sigma(n)$. Moreover, if an identity holds in every $Q \in \mathbf{P}_{r,m}(n)$, then it belongs to $\Sigma(n)$.

Denote by $\Sigma(n)^*$ the variety of n-semigroups determined by $\Sigma(n)$. We will show that $\mathbf{P}_{r,m}(n)$ is a proper subclass of $\Sigma(n)^*$ and this will imply that $\mathbf{P}_{r,m}(n)$ is not a variety.

Let i and j be nonnegative integers such that

$$r + j \equiv 1 \pmod{n},\ i + 1 + m \equiv 0 \pmod{n},$$

and let $A = \{a_0, \dots, a_{i+j}, b, b_0, \dots, b_j,\ c_1, \dots, c_n\}$ be a set with $n + i + 2j + 3$ distinct elements. Denote by F the n-semigroup which is freely generated by A in the variety $\Sigma(n)^*$, and let ρ be the minimal congruence on F such that:

$$[b_0 \dots b_j\ b^{r-1}]\ \rho\ [a_i \dots a_{i+j}\ (bc_1 \dots c_n)^{r-1}]. \tag{3.1}$$

Namely, ρ is the transitive extension of β defined by:

$$u,\ v \in F \Rightarrow (u \beta v \Leftrightarrow u \alpha v \text{ or } u = v \text{ or } v \alpha u),$$

where $u \alpha v$ iff u and v are such that:

$$u = [d_1 \dots d_{i-1}\ b_0 \dots\ b_j\ b^{r-1}\ d_i \dots d_{sn}]$$

$$v = [d_1 \dots d_{i-1}\ a_i \dots a_{i+j}\ (bc_1 \dots c_n)^{r-1}\ d_i \dots d_{sn}]$$

for some $d_1, \dots, d_{sn}$ *and* $1 \leqslant i \leqslant sn$.

We shall show that it is not true that:

$$[a_0 \dots a_{i-1}\ b_0 \dots\ b_j\ b^{r+m}\ c_1 \dots c_n]\ \rho\ [a_0 \dots\ a_{i+j}\ (bc_1 \dots\ c_n)^{m+r}]. \tag{3.2}$$

To prove that, denote by u the left hand side of (3.2), and by v the right one. By a finite number of applications of equalities that hold in F we obtain that

$$u = [a_0 \dots a_{i-1}\ b_0 \dots b_j\ b^{r+m+tp}\ c_1 \dots c_n],$$

where p is the least common multiple of m and n, and t is an abritrary nonnegative integer; u can not be written as a „product“

$$[a_0 \dots a_{i-1}\ b_0 \dots b_j\ b^r\ c_1 \dots c_n],$$

for $i + j + r \not\equiv 0 \pmod n$. Therefore there is not a w such that $w \alpha u$, and $u \alpha v_1$ iff

$$v_1 = [a_0 \dots a_{i+j}\ (bc_1 \dots c_n)^{r-1}\ b^{1+m+tp}\ c_1 \dots c_n]. \tag{3·3}$$

If v_1 is defined by (3.3) then there is not a w such that $v_1 \alpha w$. Thus we get the following statement:

$$u \beta v_1 \text{ and } v_1 \beta v_2 \Rightarrow v_2 = u \text{ or } v_1 = v_2 \text{ or } u = v_1,$$

and therefore there is not a sequence $v_1, \dots, v_k$ such that

$$u \beta v_1 \beta v_2 \beta \dots \beta v_k \beta v,$$

and this finaly implies that (3.2) does not hold.

Denote by Q the n-semigroup F/ρ, which obviously belongs to $\Sigma(n)^*$. We will show that Q does not belong to $\mathbf{P}_{r,m}(n)$, and this will imply that $\mathbf{P}_{r,m}(n)$ is a proper subclass of $\Sigma(n)^*$, i. e. that $\mathbf{P}_{r,m}(n)$ is not *a* variety.

First, we can assume that $A \subset P$, and thus we have:

$$[b_0 \dots b_j\ b^{r-1}] = [a_i \dots a_{i+j}\ (bc_1 \dots c_n)^{r-1}]. \tag{3.1'}$$

The fact that (3.2) does not hold implies that the following inequality is satisfied in Q:

$$[a_0 \dots a_{i+j}\ (bc_1 \dots c_n)^{r+m}] \neq [a_0 \dots a_{i-1}\ b_0 \dots b_j\ b^{r+m}\ c_1 \dots c_n]. \tag{3·2'}$$

If Q were an n-subsemigroup of *a* semigroup $S \in \mathbf{P}_{r,m}$, then we would have:

$$\begin{aligned}[a_0 \dots a_{i+j}\ (bc_1 \dots c_n)^{r+m}] &= a_0 \dots a_{i-1}\ [a_i \dots a_{i+j}\ (bc_1 \dots c_n)^{r-1}]\ bc_1 \dots c_n \\ &= a_0 \dots a_{i-1}\ [b_0 \dots b_j\ b^{r-1}]\ bc_1 \dots c_n \\ &= [a_0 \dots a_{i-1}\ b_0 \dots b_j\ b^{r+m}\ c_1 \dots c_n].\end{aligned}$$

This completes the proof of the following proposition:

3.1. *If* n *is not a divisor of* m *and* $r \neq 0{,}1$, *then* $\mathbf{P}_{r,m}(n)$ *is not a variety.*

4. Theorem **2** is a consequence from the following statement:

4.1. *An n-semigroup Q belongs to* $\mathbf{C}_{r,m}(n)$ *iff the following identity is satisfied in* Q:

$$\begin{bmatrix} i_1 & i_2 & & i_k \\ x_1 & x_2 & \dots & x_k \end{bmatrix} = \begin{bmatrix} j_1 & j_2 & & j_k \\ x_1 & x_2 & \dots & x_k \end{bmatrix} \tag{4.1}$$

for every sequence $i_1, \dots i_k, j_1 \dots, j_k$ *of positive integers, such that;*

$$\begin{aligned} & i_v < r \text{ or } j_v < r \Rightarrow i_v = j_v \\ & i_v \geqslant r \text{ and } j_v \geqslant r \Rightarrow i_v \equiv j_v \pmod m, \end{aligned} \tag{4.2}$$

$$\begin{aligned} i_1 + \dots + i_k &\equiv j_1 + \dots + j_k \\ &\equiv 1 \qquad \pmod n. \end{aligned} \tag{4.3}$$

Proof. 1) It is easy to see that every identity in the variety $\mathbf{C}_{r,m}$ has a form

$$\overset{i_1}{x_1} \cdots \overset{i_k}{x_k} = \overset{j_1}{x_1} \cdots \overset{j_k}{x_k}, \tag{.1'}$$

where i_v, j_v are such that (4.2) is satisfied. This implies that every identity which holds in $\mathbf{C}_{r,m}(n)$ has a form (4.1), where (4.2) and (4.3) are satisfied.

We have to show that if an n-semigroup Q satisfy all the identities (4.1) then $Q \in \mathbf{C}_{r,m}(n)$. If n is a divisor of m or $r \in \{0,1\}$, then this conclusion follows from **1.2**, **2.4** and **2.4'**. Further on, it will be assumed that n is not a divisor of m and $r > 1$.

2) Let $F \in \mathbf{C}_{r,m}$ be treely generated (in $\mathbf{C}_{r,m}$) by the carrier of the given n-semigroup Q. If $a_1, \dots a_k$ are different elements of Q and if $i_1 \dots, i_k$ are positive integers less than r, then

$$u = \overset{i_1}{a_1} \cdots \overset{i_k}{a_k}$$

is said to be an irreducible element of F, for u can be represented in a unique way as a product or powers of different elements of Q, And, $v \in F$ iz reducible if it is not irreducible, i. e. if there exists a $b \in Q$ and a positive integer j such that $v = b^{jm} v$.

Define a relation α in F by:

$$a = [a_0 \dots a_{kn}] \text{ in } Q \Rightarrow au \ \alpha \ a_0 \dots a_{kn} u,$$

where $u \in F$ or u is an empty symbol. If β is the symmetric extension of β, and τ the transitive extension of β, then τ is a congruence on F. We will show that

$$a, \ b \in Q \Rightarrow (a \tau b \Rightarrow a = b), \tag{4.4}$$

and this will complete the proof.

3) Let $a \in Q$, $u, v \in F$. Having in mind the assumptions on r, m and, n, we conclude that the following statements are statisfied.

$$(i)\ u \alpha a \Rightarrow a = u; \quad (ii)\ a \beta u \Leftrightarrow a \alpha u;$$

$$\text{(iii) } a \alpha u \text{ iff there exist } a_0, \ldots, a_{sn} \in Q \text{ such that}$$

$$a = [a_0 \ldots a_{sn}] \text{ and } u = a_0 \ldots a_{sn};$$

$$(iv)\ u \text{ is irreducible} \Rightarrow (a \alpha u \beta v \Rightarrow a \alpha v).$$

4) Assume now that $a \in Q$ and $a \alpha u_1 \beta u_2 \beta \ldots \beta u_{q-1} \beta u_q$, where u_1, u_q are reducible and $u_2, \ldots, u_{q-1}$ are irreducible. Then there exist nonnegative iniegers $k_1, \ldots, k_q$, and $c, d, a_\nu, a_{\nu\lambda} \in Q$ such that:

$$u_1 = ca_1 \ldots a_{sn} = ca_{12} \ldots a_{ik_1},\ a = [ca_1 \ldots a_{sn}]$$

$$u_2 = a_{21} \ldots a_{2k_2}, \ldots, u_q = da_{q2} \ldots a_{qk_q}$$

$$k_1 \equiv sn + 1 \pmod{m},\ k_1 \equiv k_2 \ldots \equiv k_q \pmod{n}.$$

From the reducibility of u_1 and u_q it follows that we may assume that

$$u_1 = c^{jm}\ u_1,\ u_q = d^{jm}\ u_q$$

for every $j \geqslant 0$. If $i \geqslant 0$ is such that $im + k_1 \equiv 1 \pmod{n}$, then we have:

$$\begin{aligned}
[d^{im}\ d\ a_{q2} \ldots a_{qk_q}] &= [d^{im}\ a_{q-11} \ldots a_{q-1\,k_{q-1}}] = \ldots = [d^{im}\ ca_{12} \ldots\ a_{1k_1}] \\
&= [d^{im}\ c^{rnm}\ ca_{12} \ldots a_{1k_1}] = \ldots = [d^{im}\ c^{rnm}\ da_{q2} \ldots a_{qk_q}] \\
&= [c^{rnm+im}\ da_{q2} \ldots a_{qk_q}] = \ldots = [c^{rnm+im}\ ca_{12} \ldots a_{1k_1}] \\
&= [c^{rnm}\ ca_1 \ldots a_{sn}] = [ca_1 \ldots a_{sn}] \\
&= a,
\end{aligned}$$

and this implies that $a \alpha u_q$.

5) Now, it can be casily shown the statement (4.4), and this will complete the proof.

Let $a, b \in Q$ be such that $a \tau b$. Then, there exist $u_1, \ldots, u_p$ such that $a \beta u_1\ \beta u_2\ \beta \ldots \beta\ u_p\ \beta b$. If $p = 0$ or $p = 1$, then by 3) we have $a = b$. Assume that $p \geqslant 2$. If u_1 is irreducible, then also by 3) we have $a \alpha u_2$. Thus we may assume that u_1 and u_p are reducible, and if $q \geqslant 2$ is the least integer such that u_q is reducible, then by 4) we get $a \alpha u_q$.

5. The Varietties $\mathbf{P}_{r,sn}(n)$, $\mathbf{P}_{1,m}(n)$, $\mathbf{P}_{0,m}(n)$ and $\mathbf{C}_{r,m}(n)$ are described in **1.1, 2 3, 2,4′** and **4.1** respectively. But each of these varieties is characterized by an infinite number of identities.

Clearly, every identity of the form (1.1) is a consequence from the finite set of identities where the following relations are assumed:

$$0 \leqslant i \leqslant n,\ i \leqslant p \leqslant i+n,\ p < q \leqslant n+p$$

$$p + r\ (q\text{-}p) \equiv 1 \pmod{n} \tag{5.1}$$

Thus we have the following description of $\mathbf{P}_{r,sm}(n)$.

5.1. *$Q \in \mathbf{P}_{r,sn}(n)$ iff for any integers i, p, q which satisfy* (5.1) *the identity* (1.1) *holds in Q.*

Let i, j, d, n_1, m_1 be as in **3.**, and let the integers p, q, s, k satisfy the following relations:

$$0 \leqslant p \leqslant d,\ 0 \leqslant s \leqslant 2,\ 1 \leqslant k \leqslant d, \tag{5.2}$$

$$0 \leqslant q \leqslant d,\ q+k \leqslant sd+1 \leqslant d+q+k.$$

The following two statements are corollaries from **5.1, 2.1 (2.1′),** and **2.3 (2.3′).**

5.2 (5.2′) *$Q \in \mathbf{P}_{1m}(n)$ ($Q \in \mathbf{P}_{0,m}(n)$) iff for any integers which satisfy* (5.2), *the identities* (2.3)—(2.5) ((2.3′)—(2.5′)) *hold in Q.*

It can be shown in the same way that each of the varieties $\mathbf{C}_{r,sn}(n)$, $\mathbf{C}_{1,m}(n)$, $\mathbf{C}_{0,m}(n)$ is finitely axiomatizable, but we will prove directly that each variety $\mathbf{C}_{r,m}(n)$ is finitely axiomatizable.

5. . *Let p and q be the least nonnegative integers such that*

$$p + r \equiv q + 2r + m \equiv 1 \ (mod\ n), \tag{5.3}$$

end let $n = td$, where d is the greatest common divisor of m and n. Then: $Q \in \mathbf{C}_{r,m}(n)$ iff Q is a commutative n-semigroup which satisfies the following identities:

$$[x^r\ x_1 \ldots x_p] = [x^{r+tm}\ x_1 \ldots x_p] \tag{5.4}$$

$$[x^r\ y^{r+m}\ x_1 \ldots x_q] = [x^{r+m}\ y^r\ x_1 \ldots x_q]. \tag{5.5}$$

Proof. We have to show that if a commutative n-semigroup Q satisfies the identities (5.4) and (5.5), then it satisfies all the identities (4.1).

Assume that the nonnegative integers $i_1, \ldots, i_k, j_1, \ldots, j_k$ satisfy (4.2) and (4.3). By (5.4) and (5.5) it can be easilv shown that if $i_2 = j_2, \ldots, i_k = j_k$ or $i_1 = j_2,\ i_2 = j_1,\ i_2 = j_3, \ldots,\ t_k = j_k$ then (4.1) holds. By applications of these results we obtain that all identities (4.1) hold in Q.

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n- ПОТПОЛУГРУПИ НА ПОЛУГРУПИ ШТО ГО ЗАДОВОЛУВААТ ЗАКОНОТ $x^r = x^{r+m}$

Резиме

За подмножеството Q од една полуѓрупа S велиме дека е *n-йоӣйолуѓруӣа* ако $Q^{n+1} \subseteq S$. Ако **C** е класа полугрупи, тогаш со **C** (n) ја означуваме класата n-полугрупи што можат да се сместат во **C**-полугрупи. Во трудов, имено, се проучува класата $P_{r,m}$ (n), при што $P_{r,m}$ е многукратноста полугрупи спомената во насловот. Докажуваме дека $P_{r,m}$ (n) е многукратност ако и само ако $r \in \{0,1\}$ или n е делител на m. Исто така, покажуваме дека $\mathbf{C}_{r,m}$ (n) е во секој случај многукратност, при што $\mathbf{C}_{r,m}$ е многукратноста комутативни $\mathbf{P}_{rm}$-полугрупи.