

ON A REPRESENTATION OF ALGEBRAS IN SEMIGROUPS

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0. Preliminary definitions and results. Necessary definitions and main results will be stated first.

0.1. Definitions. Let F be a nonempty set of finitary operators, such that $F_0 \cup F_1 = \emptyset$. (F_n is the set of n -ary operators in F). If A is a set and each n -ary operator f is interpreted as an n -ary operation on A , then $\mathbf{A} = (A; F)$ is said to be an F -algebra. Let $S = (S; \cdot)$ be a semigroup and $\xi : A \rightarrow S$ such a mapping that

$$\xi(fx_1 \dots x_n) = \xi(x_1) \dots \xi(x_n) \quad (0.1)$$

for each n -ary operator $f \in F_n$ and all $x_1, \dots, x_n \in A$. Then we say that $\xi : A \rightarrow S$ is a semigroup homomorphism. The notion of universal semigroup homomorphism is defined in the usual manner. If $\lambda : \mathbf{A} \rightarrow \mathbf{A}^\wedge$ is the universal semigroup homomorphism, then \mathbf{A}^\wedge is called the universal semigroup for \mathbf{A} . The cardinal number $|\lambda(A)|$ of the set $\lambda(A)$ is called the semigroup order (i.e. s -order) of \mathbf{A} , and it is denoted by $\|\mathbf{A}\|$. The algebra \mathbf{A} is said to be s -finite (s -infinite) iff $\|\mathbf{A}\|$ is finite (infinite); if $\|\mathbf{A}\| = 1$ then we say that \mathbf{A} is s -singular. If the universal semigroup homomorphism $\lambda : \mathbf{A} \rightarrow \mathbf{A}^\wedge$ is a monomorphism, then \mathbf{A} is said to be a semigroup F -algebra.

An algebra $\mathbf{A} = (A; F)$ is said to be a weak F -associative iff for all $f \in F_n$, $g \in F_m$ and $i \in \{1, 2, \dots, n\}$ the following identities are satisfied

$$\begin{aligned} fgx_1 \dots x_{m+n-1} &= gfx_1 \dots x_{m+n-1} \\ &= fx_1 \dots x_{i-1}gx_i \dots x_{m+n-1} \end{aligned} \quad (0.2)$$

A weak F -associative is called an F -associative iff for every pair of sequences f_1, \dots, f_r , g_1, \dots, g_s such that

$$f_i \in F_{n_i+1}, g_j \in F_{m_j+1}, n_1 + \dots + n_r = m_1 + \dots + m_s$$

the following identity is satisfied:

$$f_1 \dots f_r x_0 \dots x_n = g_1 \dots g_s x_0 \dots x_n \quad (0.3)$$

Throughout the paper, by d will be denoted the greatest common divisor of the numbers in the set:

$$J = \{n-1 \mid F_n \neq \emptyset\}. \quad (0.4)$$

0.2 Main results

- (i) \mathbf{A} is s -finite iff \mathbf{A}^\wedge is finite.
- (ii) If \mathbf{A} is s -infinite then $\|\mathbf{A}\| = |\mathbf{A}^\wedge|$.
- (iii) If \mathbf{A} is s -singular, then \mathbf{A}^\wedge is the cyclic group with order d .
- (iv) If $\alpha (\neq 0)$ is a given cardinal number, then each F -algebra is a subalgebra of an F -algebra with s -order α .
- (v) If A is a set and α a cardinal such that $0 < \alpha \leq |A|$, then there is an F -algebra $(A; F)$ with s -order α .
- (vi) If $|F| \geq 2$, then the class of weak F -associatives satisfies the propositions (v).
- (vii) If the direct product of a collection of algebras is s -singular, then all algebras of the collection are s -singular. If $|F| \geq 2$ and $|I| \geq 2$, then there exist I -collections of s -singular weak F -associatives whose direct products are not s -singular.
- (viii) Every semigroup F -algebra is an F -associative, and all F -associatives are semigroup F -algebras iff $d \in J$.
- (ix) An associative is s -finite iff it is finite. If an F -associative $\mathbf{A} = (A; F)$ is infinite then $\|\mathbf{A}\| = |A|$.
- (x) An associative $(A; F)$ is s -singular iff $|A| = 1$.

λ
 φ
 \mathbf{A}^\wedge
1. Universal semigroup homomorphisms. Let $\mathbf{A} = (A; F)$ be an F -algebra. A semigroup homomorphism $\lambda : \mathbf{A} \rightarrow \mathbf{A}^\wedge$ is said to be a universal one iff for every semigroup homomorphism $\xi : \mathbf{A} \rightarrow \mathbf{S}$ there is a unique homomorphism $\varphi : \mathbf{A}^\wedge \rightarrow \mathbf{S}$ such that $\xi = \varphi \lambda$.

Clearly:

1.1. If $\lambda : \mathbf{A} \rightarrow \mathbf{A}^\wedge$ and $\lambda_1 : \mathbf{A} \rightarrow \mathbf{B}$ are universal semigroup homomorphisms, then there is a unique isomorphism $\varphi : \mathbf{A}^\wedge \rightarrow \mathbf{B}$ such that $\lambda_1 = \varphi \lambda$. ■

The existence of the universal semigroup homomorphism will be shown now.

Let $\mathbf{T} = (T; \cdot)$ be the semigroup which is freely generated by the carrier A of the algebra \mathbf{A} . Thus,

$$T = \{a_1 \dots a_n \mid a_1, \dots, a_n \in A, n \geq 1\} \quad (1.1)$$

is the set of finite sequences on A , and the operation is the usual concatenation of sequences.

Let

$$\mathbf{a} = a_1 \dots a_r, \quad \mathbf{b} = b_1 \dots b_s, \quad a_i, b_j \in A.$$

We write $a \tau_0 b$ iff there exist $c, c'_1, \dots, c'_r, c''_1, \dots, c''_s \in T$ and continued products $\Pi'_i(c'_i), \Pi''_j(c''_j)$ in A such that

$$c = c'_1 \dots c'_r = c''_1 \dots c''_s, \quad a_i = \Pi'_i(c'_i), \quad b_j = \Pi''_j(c''_j). \quad (1.2)$$

(If, for example, $c' = c'_1 \in A$, then $a_1 = \Pi'_1(c'_1) = c'_1$.)

Now we shall prove the following statement:

1.2. The transitive extension τ of τ_0 is a congruence on T , and the canonical mapping:

$$\lambda : a \rightarrow a^\tau \quad (1.3)$$

is a universal semigroup homomorphism from A into $A^\wedge = T/\tau$.

Proof. Clearly, τ_0 is a reflexive and symmetric relation in T , such that

$$u \tau_0 v \Rightarrow uv \tau_0 v w, \quad wu \tau_0 w v,$$

and this implies that τ is a congruence on T .

If $f \in F_n$, $a = fa_1 \dots a_n$ in A , then $a \tau_0 a_1 \dots a_n$ in T , i.e.

$$\begin{aligned} [\lambda(a) &= a^\tau = a_1^\tau \dots a_n^\tau \\ &= \lambda(a_1) \dots \lambda(a_n). \end{aligned}$$

Thus, $\lambda : A \rightarrow A^\wedge$ is a semigroup homomorphism.

Let $S = (S; \cdot)$ be a semigroup, and $\xi : A \rightarrow S$ a semigroup homomorphism. If (1.2) is satisfied, and if $c = e_1 \dots e_t$ ($e_i \in A$), then we have:

$$\xi(a_1) \dots \xi(a_r) = \xi(e_1) \dots \xi(e_t) = \xi(b_1) \dots \xi(b_s),$$

and this implies that

$$\varphi : (a_1, \dots, a_r)^\tau \rightarrow \xi(a_1) \dots \xi(a_r)$$

is a mapping from A^\wedge into S . Clearly, φ is a homomorphism from A^\wedge into S and it is the unique mapping such that $\xi = \varphi \lambda$. \blacksquare

1.3. Let $A = (A; F)$, $A' = (A'; F)$ be F -algebras and $\lambda : A \rightarrow A^\wedge$, $\lambda' : A' \rightarrow A'^\wedge$ the corresponding universal semigroup homomorphisms. If $\varphi : A \rightarrow A'$ is a homomorphism, then there is a unique homomorphism $\varphi^\wedge : A^\wedge \rightarrow A'^\wedge$ such that $\lambda' \varphi = \varphi^\wedge \lambda$.

Proof. Let T, T', τ and τ' be defined as in the proof of 1.2. It is easy to show that:

$$a_1 \dots a_r \tau b_1 \dots b_s \Rightarrow \varphi(a_1) \dots \varphi(a_r) \tau' \varphi(b_1) \dots \varphi(b_s),$$

and this implies that

$$\varphi^\wedge : (a_1 \dots a_r)^\tau \rightarrow (\varphi(a_1) \dots \varphi(a_r))^{\tau'}$$

is a mapping from A^\wedge into A'^\wedge . Clearly, φ is a homomorphism from \mathbf{A}^\wedge into \mathbf{A}'^\wedge , and it is the unique mapping which satisfies the equation $\lambda' \varphi = \varphi^\wedge \lambda$. ■

It is easy to show that:

1.4. \wedge is a covariant functor from the category of F -algebras into the category of semigroups. ■

1.5. If $\varphi : \mathbf{A} \rightarrow \mathbf{A}'$ is an epimorphism (isomorphism), then φ^\wedge is an epimorphism (isomorphism) too.

Proof. If φ is an epimorphism, and $(a'_1 \dots a'_r)^{\tau'}$, then

$$(a'_1 \dots a'_r)^{\tau'} = \varphi^\wedge((a_1 \dots a_r)^\tau)$$

where $a'_k = \varphi(a_k)$.

If φ is an isomorphism then:

$$\varphi^\wedge(\varphi^{-1})^\wedge = (\varphi\varphi^{-1})^\wedge = (1_{A'})^\wedge, (\varphi^{-1})^\wedge\varphi^\wedge = (1_A)^\wedge. \blacksquare$$

We notice that the statement for monomorphisms is not true. Namely the proposition 2.8 states that each algebra \mathbf{A} is a subalgebra of an s -singular algebra \mathbf{B} . Thus, if \mathbf{A} is not singular, then the embedding homomorphism $\varepsilon : \mathbf{A} \rightarrow \mathbf{B}$ is a monomorphism, but $\varepsilon^\wedge : A^\wedge \rightarrow B^\wedge$ is not a monomorphism.

In the following, \mathbf{T} , \mathbf{A} , λ and τ will have the same meaning as in the proof of 2.1. If k is a positive integer, then A_k is a subset of A^\wedge defined by:

$$A_k = \{(a_1 \dots a_k)^\tau \mid a_1, \dots, a_k \in A\}. \quad (1.4)$$

1.6. If $n \in J$ then $A^\wedge = A_1 \cup \dots \cup A_n$.

Proof. This is a consequence from the following relations:

$$n \in J \Rightarrow A_{n+1} \subseteq A_1$$

$$A^\wedge = A_1 \cup \dots \cup A_k \cup A_{k+1} \cup \dots$$

1.7. \mathbf{A} is s -finite iff \mathbf{A}^\wedge is finite. If \mathbf{A} is s -infinite then $\|\mathbf{A}\| = |A^\wedge|$.

Proof. From (1.4) and 1.6 follows that if $n \in J$ then

$$\|\mathbf{A}\| \leq |A^\wedge| \leq \|\mathbf{A}\| + \|\mathbf{A}\|^2 + \dots + \|\mathbf{A}\|^n. \blacksquare$$

The following propositions are obvious.

1.8. $\|\mathbf{A}\| = \alpha$ iff $|\xi(A)| \leq \alpha$ for every semigroup homomorphism $\xi : \mathbf{A} \rightarrow \mathbf{S}$, and $|\eta(A)| = \alpha$ for a semigroup homomorphism $\eta : \mathbf{A} \rightarrow \mathbf{S}$.

1.9. If \mathbf{A}' is a homomorphic image of \mathbf{A} , then $\|\mathbf{A}'\| \leq \|\mathbf{A}\|$.

1.10. If $F' \subseteq F$ then $\|(A; F)\| \leq \|(A; F')\|$.

2. *Semigroup-singular algebras.* Some example of s -singular algebras will be given first.

2.1. Let A be a nonempty set, 0 a fixed element of A , and $\varphi : x \rightarrow x'$ an injection from A into A . If f is an n -ary operation on A such that

$$fx'^n = x, \quad fx''x'^{n-1} = fx''^{n-1}0 = 0, \quad (2.1)$$

then the algebra $\mathbf{A} = (A; f)$ is s -singular.

Proof. If $\xi : \mathbf{A} \rightarrow \mathbf{S}$ is a semigroup homomorphism, and $a \in A$, then

$$\begin{aligned} \xi(a) &= \xi(fa'^n) = \xi(f^2 a''^n a'^{n-1}) \\ &= \xi(a'')^n \xi(a')^{n-1} = \xi(fa''^{n-1}fa'' a'^{n-1}) \\ &= \xi(fa''^{n-1}0) = \xi(0). \quad \blacksquare \end{aligned}$$

2.2. Let f, g be two distinct elements of F , such that $f \in F_n, g \in F_m$. If there is an element $e \in A$ such that

$$fxe^{n-1} = x, \quad gxe^{m-1} = e, \quad (2.2)$$

for every $x \in A$, then the algebra $(A; F)$ is s -singular.

Proof. If $\xi : \mathbf{A} \rightarrow \mathbf{S}$ is a semigroup homomorphism and $a \in A$ then we have:

$$\begin{aligned} \xi(a) &= \xi(gfae^{m+n-2}) = \xi(a)\xi(e)^{m+n-2} \\ &= \xi(fgae^{m+n-2}) = \xi(fe^n) \\ &= \xi(e). \quad \blacksquare \end{aligned}$$

2.3. Let $F = F' \cup F'', F' \cap F'' = \emptyset, F' \neq \emptyset, F'' \neq \emptyset$. If $\mathbf{G} = (G; \cdot)$ is a semigroup with a zero 0 and an identity e , and if an F -algebra $\mathbf{G}(F', F'')$ is defined by:

$$f \in F' \cap F_n \Rightarrow fx_1 \dots x_n = 0 \quad (2.3)$$

$$g \in F'' \cap F_m \Rightarrow gx_1 \dots x_m = x_1 \dots x_m,$$

then $\mathbf{G}(F', F'')$ is an s -singular weak F -associative.

Proof. It is easy to see that $\mathbf{G}(F', F'')$ is a weak F -associative. Let $\xi : \mathbf{G}(F', F'') \rightarrow \mathbf{S}$ be a semigroup homomorphism. If $f \in F_n \cap F'$, $g \in F_m \cap F''$ and $a \in G$, then we have:

$$\begin{aligned}\xi(a) &= \xi(g^{n-1} a e^{(n-1)(m-1)}) = \xi(a) \xi(e)^{(n-1)(m-1)} \\ &= \xi(f^{m-1} a e^{(n-1)(m-1)}) = \xi(0),\end{aligned}$$

and this implies that $\mathbf{G}(F', F'')$ is s -singular. \blacksquare

2.4. If A is a nonempty set, and f an n -ary operator, then there is an s -singular algebra $(A; f)$.

Proof. Let $\varphi : x \rightarrow x'$ be an injective transformation of A and 0 a fixed element of A such that

$$x' = x \text{ or } x' = 0 \Rightarrow x = 0. \quad (2.4)$$

Clearly, there is an n -ary operation on f such that (2.1) is satisfied, and if f is such an operation, then by 2.1 $(A; f)$ is s -singular. \blacksquare

2.5. If $|F| \geq 2$ and if A is a nonempty set, then there is an s -singular weak F -associative.

Proof. First, a semigroup $(A; \cdot)$ with a zero and an identity can be built, and then an s -singular weak F -associative can be obtained as in 2.3. \blacksquare

If $F = F_n = \{f\}$, then a weak associative $(A; f)$ is called an n -semigroup. It is well known that:

2.6. If $(A; f)$ is an n -semigroup, then there is a semigroup $\mathbf{B} = (B; \cdot)$ such that $A \subseteq B$ and

$$f x_1 \dots x_n = x_1 \dots x_n,$$

for all $x_1, \dots, x_n \in A$. And, if \mathbf{B} is generated by A , then \mathbf{B} is said to be a covering semigroup of $(A; f)$. ([4], p. 25). \blacksquare

As a consequence of 2.6, we obtain that the assumption $|F| \geq 2$ is essential in 2.5. Namely, we have:

2.7. An n -semigroup $(A; f)$ is singular iff $|A| = 1$. \blacksquare

Now, we shall show that the class of s -singular F -algebras is not hereditary.

2.8. Every algebra is a subalgebra of an s -singular algebra.

Proof. Let $A = (A; F)$ be an F -algebra.

(i) Let $f \in F_n$, $g \in F_m$ be two distinct elements of F , and $e \notin A$. An algebra $\mathbf{B} = (A \cup \{e\}; F)$ can be defined such that A is a subalgebra of \mathbf{B} , and (2.2) is satisfied for all $x \in B$. Then, by 2.2, \mathbf{B} is s -singular.

(ii) Let $F = F_n = \{f\}$, and let B be a set and $\varphi : x \rightarrow x'$ an injective transformation of B , such that $A \subseteq B \setminus \varphi(B)$, $0 \in B$, and (2.4) is satisfied. Then,

an algebra $\mathbf{B} = (B; f)$ can be defined in which A is a subalgebra and (2.1) is satisfied. By 2.1, \mathbf{B} is s -singular.

2.9 The class of subalgebras of s -singular weak F -associatives is a proper subclass of the class of weak F -associatives.

Proof. 1) Let A be a weak F -associative and $\lambda: A \rightarrow A^\wedge$ the universal semigroup homomorphism. In a similar way as in the proof of 4.6, it can be shown that if $a, b \in A$ and $\lambda(a) = \lambda(b)$, then there exist two sequences of operators $f_1 \dots f_r, g_1 \dots g_s$ such that:

$$f_1 \dots f_r a x_1 \dots x_n = g_1 \dots g_s b x_1 \dots x_n, \quad (2.5)$$

for all $x_1, \dots, x_n \in A$.

2) Let B be a nonempty set and A the weak F -associative which is freely generated by B . It is easy to see that if $a \in C$ and $b \in A$ are two distinct elements of A , then no equation of the form (2.5) is satisfied in A . Therefore A can not be embedded in an s -singular weak F -associative.

2.10. If A is an s -singular F -algebra, then A^\wedge is the cyclic group with d elements.

Proof. Let C_d be the cyclic group with a generator c and order d . It is easy to see that the mapping $\xi: A \rightarrow C_d$ defined by: $(\forall x) \xi(x) = c$ is a universal semigroup homomorphism. \blacksquare

3. Algebras with arbitrary semigroup orders. The main results in this part are statements 3.6 and 3.7 which are generalizations of 2.4, 2.5, 2.8, and 2.9.

$$3.1. \text{ If } F(A) = \bigcup_{n=0}^{\infty} \bigcup_{f \in F_n} f A^n \quad \text{and} \quad A^* = A \setminus F(A),$$

then

$$\|A\| \geq |A^*| + 1.$$

Proof. If $0 \notin A^*$, $S = A^* \cup \{0\}$, $(\forall x, y \in S) x \cdot y = 0$ and

$$\xi(x) = \begin{cases} x & \text{if } x \in A^* \\ 0 & \text{if } x \in A \setminus A^*, \end{cases}$$

then $\xi: A \rightarrow (S; \cdot)$ is a semigroup homomorphism such that $|\xi(A)| = 1 + |A^*|$. \blacksquare

As a consequence of 3.1 we obtain that:

3.2. If A is s -singular, then it is surjective, i.e. $F(A) = A$. \blacksquare

3.3. If $A = (A; F)$ is a subalgebra of $B = (B; F)$, then

$$\|B\| \leq \|A\| + |B \setminus A|.$$

Proof. Let $\lambda_B : \mathbf{B} \rightarrow \mathbf{B}^\wedge$ be the universal semigroup homomorphism. The restriction $\xi = \lambda_B|_A : \mathbf{A} \rightarrow \mathbf{B}^\wedge$ is a semigroup homomorphism and this implies that:

$$\begin{aligned} \|\mathbf{B}\| &= |\lambda_B(B)| \leq |\lambda_B(A)| + |\lambda_B(B \setminus A)| \\ &\leq |\xi(A)| + |B \setminus A| \leq \|\mathbf{A}\| + |B \setminus A|. \blacksquare \end{aligned}$$

3.4. Let $\mathbf{A} = (A; F)$ be an algebra, C a set disjoint with A and 0 a fixed element of C . If $B = A \cup C$ and if an algebra $\mathbf{B} = (B; F)$ is defined by:

(i) \mathbf{A} is a subalgebra of \mathbf{B} ;

(ii) $f \in F_n, (b_1, \dots, b_n) \in B^n \setminus A^n \Rightarrow fb_1 \dots b_n = 0$,

then

$$\|\mathbf{B}\| = \|\mathbf{A}\| + |C|.$$

(The algebra \mathbf{B} will be denoted by $\mathbf{A}(C)$.)

Proof. Let $\xi : \mathbf{A} \rightarrow \mathbf{S}$ be a semigroup homomorphism such that $|\xi(A)| = \|\mathbf{A}\|$, and $S \cap C = \emptyset$. Define a groupoid $\mathbf{D} = (S \cup C; *)$ such that \mathbf{S} is a subgroupoid of \mathbf{D} and

$$(x, y) \in D \times D \setminus S \times S \Rightarrow x * y = 0.$$

Clearly, \mathbf{D} is a semigroup and the mapping $\eta : B \rightarrow D$ defined by

$$\eta(x) = \begin{cases} \xi(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}$$

is a semigroup homomorphism $\eta : \mathbf{B} \rightarrow \mathbf{D}$. We also have:

$$\begin{aligned} |\eta(B)| &= |\eta(A)| + |\eta(C)| = |\xi(A)| + |C| = \\ &= \|\mathbf{A}\| + |C|, \end{aligned}$$

and this implies that $\|\mathbf{A}\| + |C| \leq \|\mathbf{B}\|$, whence by 3.3 we obtain that the equality holds. \blacksquare

It is obvious that:

3.5. If \mathbf{A} is a weak F -associative, then $\mathbf{A}(C)$ is also a weak F -associative. \blacksquare

3.6. Let $\alpha (\neq 0)$ be a cardinal number. Every algebra is a subalgebra of an algebra with semigroup order α .

Proof. Let A be an F -algebra. By 2.8, A is a subalgebra of an s -singular algebra B . Let C be a nonempty set such that $B \cap C = \emptyset$, and $\alpha = 1 + |C|$. Then A is a subalgebra of $B(C)$. By 3.4, we have:

$$\|B(C)\| = \|B\| + |C| = 1 + |C| = \alpha.$$

3.7. Let A be a set and $\alpha (\neq 0)$ a cardinal number.

(i) There exists an F -algebra $A = (A; F)$ with semigroup order α iff $\alpha \leq |A|$.

(ii) If $|F| \geq 2$ and $\alpha \leq |A|$, then there is a weak F -associative $A = (A; F)$ with semigroup order α .

Proof. (i) If $A = (A; F)$ is an algebra then $\|A\| = |\lambda(A)| \leq |A|$.

Assume that $\alpha \leq |A|$, and $A = B \cup C$, $B \cap C = \emptyset$, $1 + |C| = \alpha$. By 2.4 and 2.5 there exists an s -singular algebra $B = (B; F)$, and by 3.4 if $A = B(C) = (A; F)$, we have

$$\|A\| = \|B\| + |C| = 1 + |C| = \alpha$$

(ii) By 2.5, 3.4 and 3.5. ■

Some properties of semigroup orders of direct products will be shown now.

3.8. If $(A_i)_{i \in I}$ is a collection of F -algebras, then:

$$\prod_{i \in I} \|A_i\| \leq \| \prod_{i \in I} A_i \| \leq \prod_{i \in I} |A_i|.$$

Proof. If $(\lambda_i : A_i \rightarrow A_i^\wedge)_{i \in I}$ is the corresponding collection of universal homomorphisms, then

$$\xi = (\lambda_i)_{i \in I} : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} A_i^\wedge$$

is a semigroup homomorphism such that

$$\xi(\prod_{i \in I} A_i) = \prod_{i \in I} \lambda_i(A_i). \quad \blacksquare$$

As corollary of 3.8 we obtain the following statement.

3.9. If the direct product of a collection of algebras is s -singular, then all algebras of the collection are s -singular.

The following proposition shows that the converse is not true.

3.10. If $|F| \geq 2$ and $|I| \geq 2$, then there exists a collection $(A_i)_{i \in I}$ of s -singular weak F -associatives whose direct product is not s -singular.

Proof. Let F', F'', I' and I'' be nonempty, and

$$F = F' \cup F'', \quad I = I' \cup I'', \quad F' \cap F'' = I' \cap I'' = \emptyset.$$

If \mathbf{G} is a semigroup with a zero 0 and an identity $e (\neq 0)$, and if:

$$i \in I' \Rightarrow \mathbf{A}_i = \mathbf{G}(F', F''),$$

$$i \in I'' \Rightarrow \mathbf{A}_i = \mathbf{G}(F'', F'),$$

then $(\mathbf{A}_i)_{i \in F}$ is a collection of s -singular weak F -associatives. The direct product $A = \prod_{i \in I} \mathbf{A}_i$ is a weak F -associative which is not surjective, and thus (by 3.2) it is not s -singular. ■

4. Associatives. Here, it will be shown that nontrivial s -singular associatives do not exist (Theorem 4.8) and that the semigroup order of an infinite associative is the ordinary order of the associative (Theorem 4.11).

In the following it will always be assumed that $\mathbf{A} = (A; F)$ is an F -associative and that K is the additive semigroup of nonnegative integers generated by $J \cup \{0\}$.

The following general associative law can easily be shown.

4.1. Let $n_0, \dots, n_r, m_0, \dots, m_s \in J$ and $n \in K$ be such that

$$n_0 + \dots + n_r = n = m_0 + \dots + m_s,$$

and $f_i \in F_{n_i+1}, g_j \in F_{m_j+1}$. If p_1, \dots, p_r is a sequence of nonnegative integers such that $p_\nu \leq p_{\nu+1} < n_0 + \dots + n_\nu$, then the following identity is satisfied:

$$g_0 \dots g_s x_0 \dots x_n = f_0 x_0 \dots f_1 x_{p_1} \dots f_r x_{p_r} \dots x_n.$$

In the following, every continued product $\Pi(x_0, \dots, x_n)$ will be denoted by $(x_0 \dots x_n)$; if $n = 0$ then $\Pi(x_0) = x_0$.

The following results (in slightly different formulations) are known.

4.2. The class of semigroup F -algebras is equal to the class of F -associatives iff $d \in J$. [3]

4.3. The class of semigroup F -associatives admits homomorphic images iff $d \in J$. [2].

4.4. \mathbf{A} is surjective iff $(A^{n+1}) = A$, for each $n \in K$. [2]. ■

4.5. (i) There exist $n_1, \dots, n_k \in J$ such that every $n \in K$ has a form $n = \nu_1 n_1 + \dots + \nu_k n_k$, where $\nu_i \geq 0$. If k is the minimal number with the above property, then $\{n_1, \dots, n_k\}$ is uniquely determined and it is called the basis of K .

(ii) There is a positive integer q such that

$$K^* = \{v d \mid v \geq q\} \subseteq K,$$

and if q is the least such a positive number, then K^* is said to be the regular part of K . [1]

4.6. If $n \in K^*$, $v \in \{1, \dots, n+1\}$, $a, b \in A$ and $a \tau b$, then the following identity is satisfied in A :

$$(x_1 \dots x_{v-1} a x_v \dots x_n) = (x_1 \dots x_{v-1} b x_v \dots x_n). \quad (4.1)$$

Proof. If $\mathbf{a} = a_0 \dots a_r$, $\mathbf{b} = b_0 \dots b_s$, $a_i, b_j \in A$ and $\mathbf{a} \tau_0 \mathbf{b}$, then by 1.2, there exists an $e_0 e_1 \dots e_t \in T$ ($e_v \in A$) such that the following equations are satisfied:

$$a_0 = (e_0 \dots e_{m_0}), a_1 = (e_{m_0+1} \dots e_{m_0+m_1+1}), \dots, a_r = (\dots e_t) \quad (4.2)$$

$$b_0 = (e_0 \dots e_{p_0}), b_1 = (e_{p_0+1} \dots e_{p_0+p_1+1}), \dots, b_s = (\dots e_t),$$

and

$$p_0 + \dots + p_r + r = t = m_0 + \dots + m_s + s \quad (4.3)$$

$$p_0, \dots, p_r, m_0, \dots, m_s \in K.$$

From (4.3) it follows that $d|r \Leftrightarrow d|s$, and if this is satisfied, then $n+r, n+s \in K^*$, for $n+r, n+s \geq qd$ and $d|n+r, d|n+s$.

This implies that:

$$\begin{aligned} (x_1 \dots x_{v-1} \mathbf{a} x_v \dots x_n) &= (x_1 \dots x_{v-1} e_0 \dots e_t x_v \dots x_n) \\ &= (x_1 \dots x_{v-1} \mathbf{b} x_v \dots x_n), \end{aligned} \quad (4.4)$$

for all $x_1, \dots, x_n \in A$.

Assume now that $a, b \in A$ and $a \tau b$. Then there exist

$$\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r \in T,$$

such that

$$a \tau_0 \mathbf{c}_1, \mathbf{c}_1 \tau_0 \mathbf{c}_2, \dots, \mathbf{c}_r \tau_0 b,$$

and this, by (4.4), implies that:

$$\begin{aligned}
 (x_1 \dots x_{\nu-1} a x_{\nu} \dots x_n) &= (x_1 \dots x_{\nu-1} c_1 x_{\nu} \dots x_n) \\
 &\quad \text{-----} \\
 &= (x_1 \dots x_{\nu-1} c_{\nu} x_{\nu} \dots x_n) \\
 &= (x_1 \dots x_{\nu-1} b x_{\nu} \dots x_n). \blacksquare
 \end{aligned}$$

4.7. If \mathbf{A} is surjective, and $a, b \in A$, $a \tau b$, then (4.1) is an identity in \mathbf{A} , for each $n \in K$, and $\nu \in \{1, \dots, n+1\}$.

Proof. This is a consequence from 4.4 and 4.6. \blacksquare

4.8. If \mathbf{A} is s -singular then $|A| = 1$.

Proof. By 3.2, \mathbf{A} is surjective, and by 4.7 the following identity is satisfied

$$(x_1 \dots x_n) = (y_1 \dots y_n) = c_n$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in A$ and $n \in K$.

If $m, n \in K$, then

$$c_n = ((y_0 \dots y_m) x_1 \dots x_n) = (y_0 \dots y_{m-1} (y_m x_1 \dots x_n)) = c_m (= c).$$

If we define a binary operation on A by $(\forall x, y) x \cdot y = c$, then we obtain a semigroup $(A; \cdot)$ and the identity mapping 1_A is a semigroup homomorphism from \mathbf{A} in $(A; \cdot)$. This implies that \mathbf{A} is both s -singular and semigroup F -associative, and this is possible iff $|A| = 1$. \blacksquare

4.9. Let $\{m_1, \dots, m_s\} = Q$ be the basis of the regular part K^* of K , and $\{n_1, \dots, n_p\} = P = K \setminus K^*$.

If $||\mathbf{A}|| = \alpha$ then

$$|A| \leq (1 - \alpha) [1 + (\alpha - 1)^{n_1} + \dots + (\alpha - 1)^{n_p}] + \alpha(\alpha^{m_1} + \dots + \alpha^{m_s}). \quad (4.5)$$

Proof. By 4.6, if $m \in K^*$, $\nu \in \{1, \dots, m+1\}$, and if $a, b \in A$ are such that $a \tau b$, then

$$(x_1 \dots x_{\nu-1} a x_{\nu} \dots x_m) = (x_1 \dots x_{\nu-1} b x_{\nu} \dots x_m)$$

This implies that $|(A^{m+1})| \leq \alpha^{m+1}$ and thus

$$B = \bigcup_{m \in K^*} (A^{m+1}) \Rightarrow |B| \leq \alpha \sum_{m \in Q} \alpha^m. \quad (4.6)$$

By 3.1, we have:

$$A^* = A \setminus F(A) \Rightarrow |A^*| \leq \alpha - 1. \quad (4.7)$$

Let $C = F(A) \setminus B$. If $c \in C$, and if n is the maximal number of P such that $c \in (A^{n+1})$, then there exist $a_0, \dots, a_n \in A^*$, such that $c = (a_0 \dots a_n)$, and this, by (4.7), implies that

$$|C| \leq (\alpha - 1) \sum_{n \in P} (\alpha - 1)^n. \quad (4.8)$$

Finally, (4.6), (4.7), (4.8), and $A = A^* \cup B \cup C$ imply that (4.5) is satisfied. ■

The following two statements are direct consequences from 4.9.

4.10. A is s -finite iff it is finite.

4.11. If A is infinite then $||A|| = |A|$.

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РЕЗИМЕ

ЕДНО ПРЕТСТАВУВАЊЕ НА АЛГЕБРИ ВО ПОЛУГРУПИ

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Нека $A = (A; F)$ е алгебра со носител A и множество оператори F и нека $S = (S; \cdot)$ е полугрупа. За пресликувањето $\xi: A \rightarrow S$ велме дека е полугрупен хомоморфизам, ако (0.1) е идентитет за секој n -арен оператор $f \in F$. Поимот за универзален полугрупен хомоморфизам $\lambda: A \rightarrow A^\wedge$ се воведува на обичен начин, и притоа за A^\wedge велме дека е универзална полугрупа придружена на A . Кардиналниот број $|\lambda(A)| = ||A||$ се вика полугрупен ред на A ; ако $e|A| = 1$, тогаш за алгебрата A велме дека е сингуларна. Алгебрата A се вика слаба социјатива ако еден сложен производ не зависи од распоредот на операторите; A се вика социјатива ако сложените производи зависат само од низите елементи од носителот на алгебрата, но не и од низите оператори.

Ќе формулираме неколку резултати што се докажани во работава. (i) $||A||$ е конечен ако $|A^\wedge|$ е конечен; (ii) ако $||A||$ е бесконечен, тогаш $||A|| = |A^\wedge|$; (iii) ако A е сингуларна алгебра, тогаш A^\wedge е циклична група со ред d , каде што d е најголемиот заеднички делител на броевите од множеството (0.4); (iv) ако $\alpha (\neq 0)$ е даден кардинален број, тогаш секоја алгебра е подалгебра на некоја алгебра со полугрупен ред α ; (v) ако е $|A| = \alpha$ и $0 < \beta \leq \alpha$, тогаш постои алгебра $(A; F)$ со полугрупен ред β ; (vi) ако е $|F| \geq 2$, тогаш (iv) и (v) важат за класата слаби F -асоцијативи; (vii) ако директниот производ на една колекција алгебри е сингуларен, тогаш секоја алгебра од колекцијата е сингуларна, но ако е $|F| \geq 2$, обратното не мора да важи; (viii) еден асоцијатив има конечен полугрупен ред ако е конечен; (ix) полугрупиот ред на еден бесконечен асоцијатив е еднаков со редот на асоцијативот; (x) не постои нетривијален сингуларен асоцијатив.

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