ON SOME AXIOM SYSTEMS FOR n-GROUPS

Naum Celakoski

ABSTRACT. The subject of this paper are systems of axioms for n-groups. Some necessary definitions and results are given in the first section. The main result in the second section is the theorem **2.2**: „A covering of an n-semigroup Q is a group if and only if Q is an n-group", which makes possible to prove most of the results of the third section and also to give shorter proofs of some known results. The main results in the third part are the theorems **3.2** and **3.5**, which generalize and (in a sence) improve the respective results of [4], [5] and [6].

1. PRELIMINARIES

An *n-semigroup* is an algebra $(Q, [\ ])$ with an associative n-ary operation $[\ ] : (x_1, \ldots, x_n) \to [x_1 \ldots x_n]$. By the associative law, $[\ ]$ induces an associative $k(n-1)+1$-ary operation

$$[\ ]_k : (x_1, \ldots, x_{k(n-1)+1}) \to [x_1 \ldots x_{k(n-1)+1}],$$

i.e. a $k(n-1)+1$-semigroup. As usual we shall write $[\ ]$ instead of $[\ ]_k$ and the symbols a^n, B^n, $A_1 \ldots A_n$, $B^{i-1}\, a\, B^{n-i}$, for any element $a \in Q$ and non-empty subsets $B, A_1, \ldots, A_n$ of Q, will have the usual meanings.

Any semigroup $S = S(\,.\,)$ can be considered as an n-semigroup for any integer $n \geqslant 2$. A semigroup S is called a *covering* of an n-semigroup Q iff[1] Q is an n-subsemigroup of S (i.e. $Q \subseteq S$, $Q^n \subseteq Q$) and the set Q generates S. A covering M of an n-semigroup Q is said to be *maximal* one iff every covering of Q is a homomorphic image of M. Any two maximal coverings of an n-semigroup are isomorphic.

It is well-known that for any n-semigroup Q there do exists a covering semigroup ([3], [9]). Namely, let $U = U_Q$ be the semigroup which is freely generated by the set Q, i.e. U is the set of all finite sequences, over Q,

$$U = \{(a_1, \ldots, a_k) \mid k \in \mathbf{N},\ \ a_\nu \in Q\}$$

[1] „iff" stands for „if and only if".

and the operation is the concatenation of sequences. Two elements $a = (a_1, \ldots a_i)$ and $b = (b_1, \ldots, b_j)$ of U (where $a_\nu, b_\varkappa \in Q$) are said to be *strongly linked* (in Q), shortly $a\ sl\ b$, iff there exists an element $e = (e_1, \ldots, e_t)$ of U ($e_\nu \in Q$) and two sequences of nonnegative integers $k_1, \ldots, k_i$ and $m_1, \ldots, m_j$, such that

$$\begin{aligned} a_1 &= [e_1 \ldots e_{k_1}], \quad a_2 = [e_{k_1+1} \ldots e_{k_2}], \ldots, \quad a_i = [\ldots e_t], \\ b_1 &= [e_1 \ldots e_{m_1}], \quad b_2 = [e_{m_1+1} \ldots e_{m_2}], \ldots, \quad b_j = [\ldots e_t]. \end{aligned} \tag{1.1}$$

The transitive extension l of sl is a congruence on the semigroup U.

The mapping $\lambda: Q \to U/l$ defined by $\lambda(a) = a^l$ is a homomorphism, and its restriction $\lambda_1: Q \to Q^l = \{a^l \mid a \in Q\}$ is an epimorphism. Moreover, λ is a monomorphism, i.e. λ_1 is an isomorphism, which means that the n-semigroup Q can be considered as an n-subsemigroup of the semigroup U/l. Clearly, Q^l generates U/l and so U/l is a covering of the n-semigroup Q.

The semigroup U/l is a maximal covering of Q. It is also called the *free covering* (or *universal semigroup*) for the n-semigroup Q and it is denoted by $Q^\wedge$. Since Q and Q^l are isomorphic, it is convenient to identify the l-class a^l with its representative a, i.e. to assume that $Q^l = Q$. Writting the elements of $Q^\wedge$ in the form $a_1 \ldots a_i$ instead of $(a_1, \ldots, a_i)^l$ and putting

$$Q_1 = Q, \quad Q_m = \{a_1 \ldots a_m \mid a_\nu \in Q\}, \qquad m = 2, 3, \ldots,$$

we get

$$Q_{k(n-1)+r} \subseteq Q_r \qquad (1 \leqslant r < n) \tag{1.2}$$

so that the free covering $Q^\wedge$ of the n-semigroup Q obtains the following form:

$$Q^\wedge = Q_1 \cup Q_2 \cup \ldots \cup Q_{n-1} \tag{1.3}$$

where $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

2. COVERINGS OF n-GROUPS

An n-semigroup Q is called an *n-group* iff for any sequence $a_1, \ldots, a_{n-1}, b \in Q$ there exist elements $x, y \in Q$, such that

$$[xa_1 \ldots a_{n-1}] = b, \quad [a_1 \ldots a_{n-1}\, y] = b, \tag{2.1}$$

i.e. for any sequence $a_1, \ldots, a_{n-1} \in Q$

$$Qa_1 \ldots a_{n-1} = Q = a_1 \ldots a_{n-1}\, Q. \tag{2.2}$$

If an n-semigroup Q is an n-group, then the free covering semigroup $Q^\wedge$ is a group ([2], [3] and [9]). We shall show that the converse is also true. First we shall prove the following:

2.1. Lemma. *An n-semigroup Q is an n-group* iff *Q is a $k(n-1)+1$-group for any positive integer k.*

Proof. Let the n-semigroup Q be a $k(n-1)+1$-group for some k and let $[xa_1 \dots a_{n-1}] = b$ be any equation on x in Q. Since any element of a $k(n-1)+1$-group, by (2.2), can be represented as a (non-trivial) product, we can put $a_{n-1} = [c_0 c_1 \dots c_{k(n-1)}]$ and then, putting $d = [c_0 c_1 \dots c_{n-1}]$, we get the equation

$$[xa_1 \dots a_{n-2}\, dc_n \dots c_{k(n-1)}] = b,$$

which, as an equation on x in the $k(n-1)+1$-group Q has a solution in Q. Symmetrically, any equation $[a_1 \dots a_{n-1}\, y] = b$ on y has a solution in Q Therefore, the n-semigroup Q is an n-group.

The direct part of the lemma is obvious. ∎

2.2. Theorem. *A covering G of an n-semigroup Q is a group iff Q is an n-group.*

Proof. If Q is an n-group, then $Q^\wedge$ (as we mentioned before **2.1)** is a group. Therefore, if G is a semigroup which is a covering of Q, then G, as a homomorphic image of the group $Q^\wedge$, is a group.

Conversely, let G be a group and a covering of the n-semigroup Q. Since the set Q generates G and Q is an n-subsemigroup of G, we may write

$$G = Q_1 \cup Q_2 \cup \dots \cup Q_{n-1}$$

where $Q_1 = Q$, $Q_i = \{a_1 \dots a_i \mid a_\nu \in Q\}$. Denote by $m \geqslant 2$ the smallest positive integer such that $Q^m \subseteq Q$, i.e. Q is an m-subsemigroup of G. Then

$$G = Q_1 \cup Q_2 \cup \dots \cup Q_{m-1},$$

and so the identity $e \in G$ belongs to some of the sets Q_i, $1 \leqslant i \leqslant m-1$. Let $e \in Q_p$ and let $x_1, \dots, x_{m-p} \in Q$. Then

$$x_1 \dots x_{m-p} = x_1 \dots x_{m-p}\; e = x_1 \dots x_{m-p}\, e_1 \dots e_p \in Q_m \subseteq Q,$$

i.e. $Q^{m-p} \subseteq Q$. Since m is the smallest integer such that Q is an m-subsemigroup of G, it follows that $p = m-1$, i.e. $e \in Q_{m-1}$.

Let $x \in Q$. Then the inverse x^{-1} in G is an element of Q_j for some $j : 1 \leqslant j \leqslant m-1$. Since $e \in Q_{m-1}$ and $e = xx^{-1} \in Q\, Q_j = Q_{j+1}$, it follows that $j = m-2$, i.e. $x^{-1} \in Q_{m-2}$.

We shall show that Q is an m-group. Let $xa_1 \ldots a_{m-1} = b$ be an equation on x in Q. As an equation in the group G, it has a solution x:

$$x = ba_{m-1}^{-1} \ldots a_1^{-1} \in QQ_{(m-2)(m-1)} = Q_{(m-2)(m-1)+1} \subseteq Q.$$

Symmetrically, any equation $a_1 \ldots a_{m-1} y = b$ has a solution $y \in Q$. Therefore Q is an m-group.

Let $n - 1 = q(m - 1) + r$, $0 \leqslant r < m - 1$. Since Q is an m-group, it follows that $Q^m = Q$ and, more generally, $Q^{k(m-1)+1} = Q$ for any $k \in \mathbf{N}$. If were $r > 0$, then, according to the assumption that Q is an n-semigroup, we would have

$$Q \supseteq Q^n = Q^{q(m-1)+r+1} = Q^{q(m-1)+1} Q^r = Q Q^r = Q^{r+1},$$

i.e. Q would be an $r+1$-semigroup, and $r + 1 < m$, which contradicts the choice of m. Therefore $r = 0$, i.e. $n - 1 = q(m - 1)$. By Lemma **2.1** it follows that Q is an n-group. ∎

As a special case of Theorem **2.2** we have the following:

2.3. Corollary. *An n-semigroup Q is an n-group iff the free covering $Q^\wedge$ is a group.* ∎

Also:

2.4. Corollary. a) *The solutions x, y of the equations* (2.1) *in an n-group are unique.*

b) *If Q is an n-group, then every equation*

$$[a_1 \ldots a_{i-1}\, x\, a_{i+1} \ldots a_n] = b$$

on x in Q has unique solution in Q for every $i = 1, \ldots, n$. ∎

Using **2.3** we shall prove the following:

2.5. Proposition. *An n-semigroup Q $(n \geqslant 3)$ is an n-group iff for some $k \in \{2, \ldots, n-1\}$ any equation* (*on x in Q*) *of the from*

$$[a_1 \ldots a_{k-1}\, x\, a_{\varkappa+1} \ldots a_n] = b \tag{2.3}$$

has a solution x in Q.

Proof. If Q is an n-group, then by **2.4** b), the equation (2.3) has unique solution x in Q. To prove the converse, we shall show that the free covering $Q^\wedge$ of the n-semigroup Q is a group. Let $ax = b$ be an equation on x in the semigroup $Q^\wedge$, where $a = a_1 \ldots a_i$, $b = b_1 \ldots b_j$ $(a_\nu, b_\varkappa \in Q)$ and choose elements $c_{i+2}, \ldots, c_{2n} \in Q$. Since the equation (2.3) is solvable, there exists an element $x_k \in Q$, such that

$$[[a_1 \ldots a_i c_{i+2} \ldots c_{n+1}]\, c_{n+2} \ldots c_{n+k-1}\, x_k\, c_{n+k+1} \ldots c_{2n}] = b_1.$$

and then, putting $x = c_{i+2} \dots c_{n+k-1} x_k c_{n+k+1} \dots c_{2n} b_2 \dots b_j$, we get $ax = b$.

Thus, for any $a, b \in Q^\wedge$, there exists an $x \in Q^\wedge$ such that $ax = b$. By symmetry, the same is true for the equation $ya = b$ on y. Therefore $Q^\wedge$ is a group and, by **2.2**, Q is an n-group. ∎

A sequence $e_1, \dots, e_{n-1}$ of an n-semigroup Q is called a left (right) identity sequense or *left* (*right*) *identity* of Q iff

$$(\forall x \in Q) \quad [e_1 \dots e_{n-1} x] = x \quad ([x e_1 \dots e_{n-1}] = x). \tag{2.4}$$

2.6. Lemma. *An n-semigroup Q has a left and right identity iff its covering semigroup* $Q^\wedge$ *has an identity. Moreover, if every* $x \in Q$ *has a left inverse* $x \in Q^\wedge$, *then* $Q^\wedge$ *is a group.*

Proof. If $e = e_1 \dots e_{n-1} \in Q^\wedge$ is a left and right identity in the n-semigroup Q, then for any element $a = a_1 \dots a_i$ of $Q^\wedge$ $(a_\nu \in Q)$

$$ea = e_1 \dots e_{n-1} a_1 \dots a_i = [e_1 \dots e_{n-1} a_1] a_2 \dots a_i = a_1 a_2 \dots a_i = a,$$

and also $ae = a$, which means that e is the identity of $Q^\wedge$. Conversely, if e is the identity of $Q^\wedge$, then $e \in Q_{n-1}$ and clearly e is a left and right identity in the n-semigroup Q.

The second part of the lemma follows by the fact that the set Q generates $Q^\wedge$. ∎

3. AXIOM SYSTEMS FOR n-GROUPS

We introduced the concept of n-group in the previous section as an algebra Q with one n-ary associative operation [ ] in which the equations of the from (2.1) are solvable on x, y respectively. Some characterizations of this concept are given by Theorem **2.2** and its corollaries, and thus they can be regarded as other definitions of n-group. We intend to state in this section some equivalent systems of axioms for n-groups. Therefore we shall reformulate some of the results of the previous section in the following theorem.

3.1. Theorem. *Let Q be an n-semigroup. The following statements are equivalent:*

(i) *Q is an n-group* (*in the sence of* (2.1)).

(ii) *Q satisfies the relations* (2.2).

(iii) *Q is a* $k(n-1)+1$-*group for any* $k \in \mathbf{N}$.

(iv) *The free covering of Q is a group.*

(v) *Some covering of Q is a group.*

If $n \geqslant 3$:

(vi) *Any equation in Q of the form* (2.3), *for some* k: $2 \leqslant k \leqslant n-1$, *has a solution x on Q.* ∎

It seems that Dörnte [1] was the first who generalized the group concept by assuming the operation to be any n-ary (for some fixed integer $n \geqslant 3$) and letting the other axioms for group: associativity of the operation and solvability of certain equations. Larger and deeper investigations of this concept was made by Post in his monograph [2], where it is used Dörnte's definition of n-group, slightly modified:

(vii) Given a class of elements Q, and an operation $[x_1 \dots x_n]$, it is said that the elements of Q constitute an *n-adic group* under [ ] if the following two conditions are satisfied:

1) If any n of the $n+1$ symbols in an equation of the form

$$[x_1 \dots x_n] = x_{n+1}$$

represent elements in Q, the remaining symbol also represents an element in Q, and is uniquely determined by this equation.

2) The elements of Q satisfy the associative law under [ ].

This definition of n-group, by **2.5**, is equivalent with (i) (which is outlined in [2], p. 213 too).

Another definition of n-group, which involves the concepts of left and right identities of an n-group instead of solvability of certain equation (it can be found in [4] and [5]) is the following:

(viii) An n-semigroup Q ($n \geqslant 3$) is an n-group iff for every sequence $x_1, \dots, x_{n-1} \in Q$ there exists an element $x' \in Q$ such that $x_1 \dots x_{n-2} x'$ is a left and $x' x_1 \dots x_{n-2}$ is a right identity of Q.

The equivalence of this definition with (i) is a consequence of the following theorem which gives an another (equivalent) definition of n-group and generalizes (viii).

3.2. Theorem. *An n-semigroup Q ($n \geqslant 3$) is an n-group iff, for some* $k \in \{1, 2, \dots, n-2\}$,

$$(\forall\, x_1, \dots, x_k \in Q)\, (\exists\, x_1', \dots, x'_{n-k-1} \in Q)\, (\forall\, y \in Q)$$

$$[x_1 \dots x_k\, x_1' \dots x'_{n-k-1}\, y] = y = [y x_1' \dots x'_{n-k-1}\, x_1 \dots x_k]. \quad (3.1)$$

Proof. Let Q be an n-group, k a fixed element of $\{1, 2, \dots, n-2\}$ and e the identity of the free covering group $Q^\wedge$. If $x_1, \dots, x_k \in Q$, then $(x_1 \dots x_k)(x_1 \dots x_k)^{-1} = e$ in $Q^\wedge$ and by the proof of Theorem **2.2**, e belongs to Q_{n-1}. Since $x_1 \dots x_k \in Q_k$, it follows that $(x_1 \dots x_k)^{-1} \in Q_{n-k-1}$, i.e. $(x_1 \dots x_k)^{-1} = x_1' \dots x'_{n-k-1}$. Obviously

$$x_1 \dots x_k x_1' \dots x'_{n-k-1} = x_1' \dots x'_{n-k-1}\, x_1 \dots x_k = e$$

is a left and right identity of Q and thus the relations (3.1) hold.

Conversely, let Q be an n-semigroup in which (3.1) hold. As in Lemma **2.6**, we see that $e = x_1 \dots x_k x_1' \dots x'_{n-k-1}$ is a left identity and $x_1' \dots x'_{n-k-1}$ $x_1 \dots x_k = e$ is a right identity of $Q^\wedge$, i.e. e is the identity of $Q^\wedge$.

Let $a = a_1 \dots a_i$ be any element of $Q^\wedge (a_\nu \in Q)$. If $i = k$, then by (3.1), there exist $a_1', \dots, a'_{n-k-1} \in Q$, such that $a' = a_1' \dots a'_{n-k-1}$ is a left inverse of a in $Q^\wedge$. If $i<k$, then we choose elements $c_1, \ldots, c_{k-i} \in Q$ and, by (3.1), there exist $a_1', \ldots, a'_{n-k-1} \in Q$ such that $a' = a_1' \ldots a'_{n-k-1}$ $c_1 \ldots c_{k-i}$ is a left inverse of a in $Q^\wedge$. If $i>k$, then we choose elements $c_1, \ldots, c_{n-i}$ of Q and put $c = [c_1 \ldots c_{n-i}\, a_1 \ldots a_i]$. By the above considerations, for this $c \in Q^\wedge$, there exists a left inverse $a_1' \ldots a'_{n-2}$ of c, so that $a' = a_1' \ldots a'_{n-2}\, c_1 \ldots c_{n-i}$ is a left invrse of a.

Thus the semigroup $Q^\wedge$ is a group and by **2.2**, Q is an n-group. ∎

Note that (3.1) in **3.2** can be substitute by:

$$[yx_1 \ldots x_k x_1' \ldots x'_{n-k-1}] = y = [x_1' \ldots x'_{n-k-1}\, x_1 \ldots x_k y]. \qquad (3.1')$$

This theorem gives another definition of n-group which we denote by (ix). It is clear that (viii) is a special case of (ix), when $k = n-2$. In this case, for any $x_1, \ldots, x_{n-2} \in Q$ there exists a unique element x' of Q, such that $x_1 \ldots x_{n-2}x'$ and $x'x_1 \ldots x_{n-2}$ are left and right identitics of Q. This determines a mapping from the cartesian $(n-2)$-nd power of the set Q into Q, which we shall denote by $(\ )^{-1}$ and the element x' which is uniquely determined by the sequence $x_1, \ldots x_{n-2}$ will be denoted by $(x_1 \ldots x_{n-2})^{-1}$. This suggests another definition of n-group ([5] and also a consequence of **3.2**):

(x) An n-semigroup Q ([ ]) is an n-group $(n \geqslant 3)$ iff there exists an $(n-2)$-ary operation $(\ )^{-1}$ on Q such that for any $x_1, \ldots, x_{n-2}, y \in Q$ the following identity equalities hold:

$$[yx_1 \ldots x_{n-2}\,(x_1 \ldots x_{n-2})^{-1}] = y = [(x_1 \ldots x_{n-2})^{-1}\, x_1 \ldots x_{n-2}\, y].$$

Another characterization of n-group as algebra with one n-ary associative operation and one unary operation is given in [6]:

(xi) An n-semigroup Q $(n \geqslant 3)$ is an n-group iff for any x of Q there exists a x' of Q, such that the following identity equalities hold:

$$[x'\, x^{n-2}\, y] = y = [yx^{n-2}\, x'], \qquad (3.2)$$

$$[xx'\, x^{n-3}y] = y = [yx^{n-3}\, x'\, x]. \qquad (3.3)$$

This theorem is proved also in [7], p. 26—29, and it is cited in [8], p. 53. We shall prove a more general result from which (xi) will follow.

(xii) **3.3 Theorem.** *An n-semigroup $Q\,(n \geqslant 3)$ is an n-group iff for every $x \in Q$ there exists an $x' \in Q$, such that for some $p: 0 \leqslant p \leqslant n-2$ and for some $s: 0 \leqslant s \leqslant n-2$ the following identity equalities hold:*

$$[x^p\, x'\, x^{n-p-2}\, y] = y = [y\, x^{n-s-2}\, x'\, x^s]. \qquad (3.4)$$

Proof. Let Q be an n-group. For any $x \in Q$, there exists an $x' \in Q$ such that $[x^{n-1}\, x'] = x$. Since

$$x^n = [[x^{n-1}\, x']\, x^{n-1}] = x^{n-i}\,[x^{i-1}\, x'\, x^{n-i}]\, x^{i-1}],$$

by cancellation, we obtain

$$[x^{i-1}\, x'\, x^{n-i}] = x \tag{3.5}$$

for any $i : 1 \leqslant i \leqslant n$. To prove the first equality of (3.4), let y be any element of Q and choose elements $z_1, \ldots, z_{n-1} \in Q$, such that $y = [xz_1 \ldots z_{n-1}]$. Then, using (3.5):

$$[x^i\, x'\, x^{n-i-2}\, y] = [[x^i\, x'\, x^{n-i-1}]\, z_1 \ldots z_{n-1}] = [x\, z_1 \ldots z_{n-1}] = y.$$

Symmetrically for the second equality (3.4).

Conversely, let Q be an n-semigroup in which the relation (3.4) hold. If x is any element of Q, then $e = x^p x' x^{n-p-2} = x^{n-s-2}\, x' x^s$ is a left and right identity of Q, i.e. the identity of the semigroup $Q^\wedge$. If $p \geqslant 1$, then $x^{p-1} x' x^{n-p-2}$ is a right invrse of x and if $p = 0$, then $x' x^{n-3}$ is a left inverse of x in $Q^\wedge$. By **2.6**, $Q^\wedge$ is a group and by **2.2**, Q is an n-group. ∎

It can be given a "direct proof" for the converse of this theorem as it is proved the theorem (xi) in [7]. However, the proof given here is considerably shorter than the direct proof (which does not include the cases when $p = 0$ and $s = 0$). This is an example of the efficiency and usefulness of Theorem **2.2**.

If $p = 0 = s$ and $p = 1 = s$ in (3.4), then we obtain (3.2) and (3.3) respectively. Thus, as concequences of **3.3** we can state the following:

3.4. Corollary. *An n-semigroup* $Q(n \geqslant 3)$ *is an n-group iff the relations* (3.3) *hold in* Q. ∎

5.5. Corollary. *An n-semigroup* $Q\,(n \geqslant 3)$ *is an n-group iff the relations* (3.2) *hold in* Q. ∎

These results show that the system of axioms for n-group in [6], where the relations (3.2) and (3.3) are included, is not independent. According to this, it can be asked whether some of the relations (3.2) (i.e. of (3.3), (3.4)) can be omited. The answer is negative. For example, if Q is an n-semigroup of right zeros with at least two elements, then for any x, y of Q the first relation of (3.2) will hold, where x' is arbitrary (for example, $x' = x$). However, Q is not an n-group. Symmetrically for the other relation (3.2) and also for (3.3) and (3.4).

Next question would be: Is it possible to substitute the relations (3.2) by some ,,weaker" relations, for example, by the condition

$$(\forall\, a \in Q)\, (\exists!\, x \in Q)\, [x\, a^{n-1}] = a = [a^{n-1}\, x] \tag{3.6}$$

such that Q is an n-group? The answer, for the cited example, is negative. Namely, if Q is an n-band (i.e. $a^n = a$ for all $a \in Q$) then (3.6) hold, but Q may not be an n-group (and this is true for $n = 2$ too). Moreover, even when the n-semigroup Q is cancellative and with the property (3.6), it can happen Q not to be an n-group; for example, (3.6) hold in any cancellative semigroup with an identity (such an example is the multiplicative semigroup of the positive integers, which is not a group).

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ЗА НЕКОИ СИСТЕМИ АКСИОМИ ЗА *n*-ГРУПИТЕ

Наум Целакоски

Резиме

Предмет на оваа работа се некои системи аксиоми за поимот *n-група*, т.е. за алгебра Q со една n-арна ($n \geqslant 2$) асоцијативна операција [ ]: $(x_1, \ldots, x_n) \to [x_1 \ldots x_n]$ (којашто се вика и *n-полугрупа*), во која секој пар равенки од обликот (2.1) има решение x, y во Q. Ако n-полугрупата Q е n-потполугрупа од некоја полугрупа S и ако S е генерирана од множеството Q, тогаш полугрупата S се вика *покривка* на n-полугрупата Q. Во првиот дел од работава се изнесува познатиот резултат дека за секоја n-полугрупа Q постои *слободна покривка* (т.е. покривка $Q^\wedge$ на Q, таква што секоја друга покривка на Q е хомоморфна слика на $Q^\wedge$). Добро е познато дека, ако една n-полугрупа Q е n-група, тогаш нејзината слободна покривка е група. Во вториот дел од работава се докажува поошта теорема:

1°. *Некоја покривка G на една n-полугрупа Q е група ако и само ако Q е n-група.*

Со помош на таа теорема се даваат докази на неколку познати карактеристични својства на n-групите, а во trtiot дел од работава, таа се користи за докажување и на некои нови резултати:

2°. *Една n-полугрупа Q ($n \geqslant 3$) е n-група ако и само ако, за некој $k \in \{1, 2, \ldots, n-2\}$ се исполнети релациите* (3.1).

(Специјално, ако во (3.1) се стави $k = n - 2$, се добива една карактеризација на n-група што е дадена во [4] и [5].)

3°. *Една n-полугрупа Q $(n \geqslant 3)$ е n-група ако и само ако за секој $x \in Q$, постои $x' \in Q$, така што за некој $p : 0 \leqslant p \leqslant n-2$ и за некој $s : 0 \leqslant s \leqslant n-2$ се исполнети идентитетите* (3.4).

Специјално, за $p = 0 = s$ и $p = 1 = s$ се добиваат идентитетите (3.2) и (3.3). Како последица од **3°** се добива дека: Една n-полугрупа Q $(n \geqslant 3)$ е n-група ако и само ако се исполнети во Q релациите (3.2) или (3.3). Според тоа, системот аксиоми за n-група, каде што се вклучени релациите (3.2) и (3.3) (во [6]) е зависен. На крајот се разгледуваат некои примери, за кои натамошна редукција на релациите (3.2) или (3.3), во таа смисла, не е можна.