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ON QUASIGROUPS

The main object of this paper is a convenient generalization of the notion of *binary quasigroups* to the *α-quasigroups* for arbitrary ordinal α. At first, in **1—4** simple generalizations of „binary" definitions and results are given. The theorem **5.1** is the main result of this paper. It seems to us that the proof of this theorem is simpler than the known proofs of the corresponding theorem for binary quasigroups. (For example, [1] I.2) Identities in the class of α-quasigroups are considered in the last part of the paper.

Special classes of α-quasigroups may be obtained by adding some new axioms; thus, α-groups may be defined as asssociative α-quasigroups. In the papers [4] and [5] it is shown that there exist only trivial ω-groups; this results sugest the question whether the same is true for every infinite ordinal α.

1. α-sequences. Let α be an ordinal number; as usual, we assume that:

$$\alpha = \{\beta \mid \beta \text{ is an ordinal such that } \beta < \alpha\}. \tag{1.1}$$

If P is a set then the elements of P^α are called *α-sequences* on P. An α-sequence on P is denoted by $\langle a_\beta : \beta \in \alpha \rangle = \mathbf{a}$; if $\gamma \in \alpha$ then we write:

$$\mathbf{a} = \langle a_\beta : \beta \in \gamma \rangle \, a_\gamma \, {}^\gamma\mathbf{a}, \tag{1.2}$$

where

$${}^\gamma\mathbf{a} = \langle a_{\gamma+1+\beta} : \beta \in \alpha - (\gamma + 1) \rangle; \tag{1.3}$$

also if $\gamma, \delta \in \alpha$ and $\gamma \neq \delta$, then by $\mathbf{a}_{\delta,\gamma}$ is denoted the α-sequence $\mathbf{b} = \langle b_\beta : \beta \in \alpha \rangle$ such that $b_\beta = a_\beta$ for $\beta \neq \gamma, \delta$ and $b_\gamma = a_\delta$, $b_\delta = a_\gamma$.

2. Partial α-quasigroups. Let P be a set, D a subset of P^α and h a mapping of D into P. Then h is said to be a *partial α-operation* in P, and D is called the domain of h. Let $\langle h_\beta : \beta \in \alpha + 1) \rangle = \mathbf{H}$ be an $\alpha + 1$-sequence of partial α-operations of P. The partial algebra $(P, \mathbf{H})$ is said to be a *partial α-quasigroup* iff:

$$b = h_\alpha \, \mathbf{a} \Leftrightarrow a_\gamma = h_\gamma \langle a_\beta : \beta \in \gamma \rangle \, b \, {}^\gamma\mathbf{a}, \tag{2.1}$$

for each $\gamma \in \alpha$, and $\mathbf{a} = \langle a_\beta : \beta \in \alpha \rangle \in D_\alpha$. ($D_\alpha$ is the domain of h_α).

It is easy to see that the following four statements hold:

2.1 Let $\mathbf{H} = \langle h_\beta : \beta \in \alpha + 1\rangle$ be an $\alpha + 1$-sequence of partial α-operations on P. The partial algebra $(P, \mathbf{H})$ is a partial α-quasigroup iff the following two statements are satisfied:

$$\mathbf{a} \in D_\alpha \Rightarrow a_\gamma = h_\gamma \langle a_\beta : \beta \in \gamma\rangle h_\alpha \mathbf{a} \ {}^\gamma\mathbf{a} \tag{2.2}$$

$$\mathbf{a} \in D_\gamma \Rightarrow a_\gamma = h_\alpha \langle a_\beta : \beta \in \gamma\rangle h_\gamma \mathbf{a} \ {}^\gamma\mathbf{a}, \tag{2.3}$$

for every $\gamma \in \alpha$, $\mathbf{a} = \langle a_\beta : \beta \in \alpha\rangle \in P^\alpha$ ∎

2.2 If $(P, \mathbf{H})$ is a partial α-quasigroup and if $\gamma, \delta \in \alpha$, $\gamma \neq \delta$ then:

$$\mathbf{a}_{\gamma,\delta} \in D_\gamma \Rightarrow a_\gamma = h_\delta \langle a_\beta : \beta \in \gamma\rangle h_\gamma \mathbf{a}_{\gamma,\delta} \ {}^\gamma\mathbf{a}. \ ∎ \tag{2.4}$$

2.3 If $(P, \mathbf{H})$ is a partial α-quasigroup then the partial α-algebra (P, h_δ) is cancellative for each $\delta \in \alpha + 1$, i. e. (for any $\gamma \in \alpha$, $\mathbf{a} = \langle a_\beta : \beta \in \alpha\rangle \in P^\alpha$, $b \in P$):

$$h_\delta \mathbf{a} = h_\delta \langle a_\beta : \beta \in \gamma\rangle b \ {}^\gamma\mathbf{a} \Rightarrow a_\gamma = b. \ ∎ \tag{2.5}$$

2.4. Let (P, h_α) be a cancellative partial α-algebra, and let for each $\gamma \in \alpha$ a partial α-operation h_γ on P be defined by (2.1). Then the obtained algebra $(P, \langle h_\gamma : \gamma \in \alpha + 1\rangle)$ is a partial α-quasigroup. ∎

3. α-quasigroups. A partial α-operation h on P is called an *α-operation* on P if P^α is the domain of h, i. e. if h is a mapping of P^α into P. A partial α-quasigroup $(P, \mathbf{H})$ is called an *α-quasigroup* if h_γ is an α-operation on P for each $\gamma \in \alpha + 1$. By **2.1** and **2.2** we obtain:

3.1 Let $\mathbf{H} = \langle h_\beta : \beta \in \alpha + 1\rangle$ be an $\alpha + 1$-sequence of α-operations on P. The α-algebra $(P, \mathbf{H})$ is an α-quasigroup iff for any $\gamma \in \alpha$ and $\mathbf{a} = \langle \alpha_\beta : \beta \in \alpha\rangle \in P^\alpha$ the following equalities hold:

$$\begin{aligned} a_\gamma &= h_\alpha \langle a_\beta : \beta \in \gamma\rangle h_\gamma \mathbf{a} \ {}^\gamma\mathbf{a} \\ &= h_\gamma \langle a_\beta : \beta \in \gamma\rangle h_\gamma \mathbf{a} \ {}^\gamma\mathbf{a}. \ ∎ \end{aligned} \tag{3.1}$$

3.2 If $(P, \mathbf{H})$ is an α-quasigroup $\mathbf{a} = \langle a_\beta : \beta \in \alpha\rangle$ and if $\gamma, \delta \in \alpha$, $\gamma \neq \delta$ then:

$$a_\gamma = h_\delta \langle a_\beta : \beta \in \gamma\rangle h_\gamma \mathbf{a}_{\gamma,\delta} \ {}^\gamma\mathbf{a}. \ ∎ \tag{3.2}$$

An α-quasigroup may be defined as an algebra with an α-operation in the following way:

3.3 Let h_α be an α-operation on P and for each $\gamma \in \alpha$ h_γ be defined by (2.2). Then the algebra $(P, \mathbf{H})$ is an α-quasigroup iff:

$$(\forall \mathbf{a} \in P^\alpha)(\forall \gamma \in \alpha)(\exists!\, x \in P)\ h_\alpha \langle a_\beta : \beta \in \gamma\rangle\, x \ {}^\gamma\mathbf{a} = \mathbf{a}_\gamma. \ ∎ \tag{3.3}$$

4. Words. Let α be an ordinal number, and ξ_α the least infinite ordinal whose cardinal is larger than the cardinal of α. Let $P(\neq\emptyset)$ and $\Omega=\{f_\beta | \beta \in \alpha+1\}$ be two disjoint sets; the elements of Ω are called *α-operators*. Denote by $W(P;\Omega)$ the set of all η-sequences on $P \cup \Omega$ where $\eta \in \xi_\alpha$. The intersection of all subsets C of $W(P;\Omega)$ which satisfy the statements:

$$P \subseteq C,$$

$$\gamma \in \alpha+1,\ \mathbf{u} = \langle u_\beta : \beta \in \alpha \rangle \in C^\alpha \Rightarrow f_\gamma \mathbf{u} \in C, \tag{4.1}$$

is said to be the *algebra of Ω-words* on P, and is denoted by Ω_P Below we give a more detailled description of Ω_P.

4.1. Let $\{B_\lambda | \lambda \in \xi_\alpha\}$ be a collection of subsets of $W(P,\Omega)$ defined in the following way:

$$B_0 = P,$$

$$B_{\lambda+1} = B_\lambda \cup \{f_\gamma \mathbf{u} \mid \gamma \in \alpha+1,\ \mathbf{u} \in B_\lambda{}^\alpha\} \tag{4.2}$$

and if $\lambda \in \xi_\alpha$ is a limite ordinal, then

$$B_\lambda = \bigcup_{\nu \in \lambda} B_\nu. \tag{4.3}$$

Then, the folllowing equation is satisfied:

$$\Omega_P = \bigcup_{\lambda \in \xi_\alpha} B_\lambda. \tag{4.4}$$

Proof. Assume that C is a subset of $W(P,\Omega)$, such that (4.1) holds. Then, we have $B_\lambda \subseteq C$ for eact $\lambda \in \xi_\alpha$ and therefore $\cup B_\lambda \subseteq \Omega_P$. Clearly the right hand side of (4.4) satisfies the statements (4.1). ∎

If $u \in B_\lambda$, and $u \notin B_\nu$ for any $\nu \in \lambda$, then we say that λ is the *range* of u and write $\lambda = r(u)$.

4.2 If $u = f_\gamma \langle u_\beta : \beta \in \alpha \rangle$, then $(\forall \beta \in \alpha)\ r(u_\beta) < r(u)$.

Proof. First, note that the range $r(u)$ of a word u is not a limite ordinal, and $r(u) = \lambda + 1 \Leftrightarrow u \in B_{\lambda+1} \setminus B_\lambda$. Therefore, if $u = f_\gamma \langle u_\beta : \beta \in \alpha \rangle$ and $r(u) = \lambda+1$, then $\langle u_\beta : \beta \in \alpha \rangle \in B_\lambda{}^\alpha$, i. e. $(\forall \beta \in \alpha)\ r(u_\beta) \leqslant \lambda < \lambda + 1$. ∎

5. Embeddings of partial α-quasigroups into α-quasigroups$_6$

If an α-quasigroup $(Q, \overline{\mathbf{H}})$ is generated by a partial α-quasigroup $(P, \mathbf{H})$ and if every homomorphism λ of $(P, \mathbf{H})$ into an α-quasigroup $(Q', \mathbf{H}')$ can be extended to a homomorphism $\varphi:(Q, \overline{\mathbf{H}}) \to (Q'\ \mathbf{H}')$, then $(Q, \overline{\mathbf{H}})$ is said to be *freely generated* by $(P, \mathbf{H})$.

The main result of this paper is the following theorem.

5.1 If $(P, \mathbf{H})$ is a partial α-quasigroup, then there is an α-quasigroup $(Q, \overline{\mathbf{H}})$ which is freely generated by $(P, \mathbf{H})$,.

Proof. 1) Let Ω_P be the algebra of Ω-words defined as in **4**. The notion of subwords is defined in the usual manner. An Ω-word $v \in \Omega_P$ is said to be *irreducible* if neither of its subwords has one of the following four forms:

$$f_\alpha \langle u_\beta : \beta \in \gamma \rangle f_\gamma \mathbf{u} \, {}^\gamma\mathbf{u} \tag{5.1}$$

$$f_\gamma \langle u_\beta : \beta \in \gamma \rangle f_\alpha \mathbf{u} \, {}^\gamma\mathbf{u} \tag{5.2}$$

$$f_\tau \langle u_\beta : \beta \in \gamma \rangle f_\gamma \mathbf{u}_{\gamma,\tau} \, {}^\gamma\mathbf{u} \tag{5.3}$$

$$f_\gamma \, \mathbf{a} \tag{5.4}$$

where $\mathbf{u} \in \Omega_P$, $\mathbf{a} \in D_\gamma \subseteq P^\alpha$, $\nu, \delta, \tau \in \alpha$ and $\gamma \neq \tau$.

2) Denote by Q_P the set of all irreducible words, and define a collection $\langle \overline{h}_\gamma : \gamma \in \alpha + 1 \rangle$ of α-operations on Q_P as follows.

Let $\mathbf{u} = \langle u_\beta : \beta \in \alpha \rangle \in Q_P^\alpha$, $\tau \in \alpha + 1$ and $v = f_\tau \, \mathbf{u}$. Then:

$$v \in Q_P \Rightarrow v = \overline{h}_\tau \, \mathbf{u} \tag{5.5}$$

$$u_\gamma = f_\gamma \langle u_\beta : \beta \in \gamma \rangle \, u \, {}^\gamma\mathbf{u} \Rightarrow u = \overline{h}_\alpha \, \mathbf{u} \tag{5.6}$$

$$u_\gamma = f_\alpha \langle u_\beta : \beta \in \gamma \rangle \, u \, {}^\gamma\mathbf{u} \Rightarrow u = \overline{h}_\gamma \, \mathbf{u} \tag{5.7}$$

$$\gamma \neq \delta, \ \mathbf{w} = \langle u_\beta : \beta \in \gamma \rangle \, u \, {}^\gamma\mathbf{u}, \ u_\gamma = f_\gamma \, \mathbf{w}_{\gamma,\delta} \Rightarrow u = \overline{h}_\delta \, \mathbf{u} \tag{5.8}$$

$$\tau \in \alpha + 1, \ \mathbf{a} \in P^\alpha, \ b = h_\tau \, \mathbf{a} \text{ in } (P, \mathbf{H}) \Rightarrow b = \overline{h}_\tau \, \mathbf{a}. \tag{5.9}$$

By **4.2** and (5.5) — (5.9), $\langle \overline{h}_\beta : \beta \in \alpha + 1 \rangle$ is a collection of α-operations on Q_P.

By a straightforward computation it can be shown that the identities (3.1) are satisfied, and this would imply that $(Q_P, \overline{\mathbf{H}})$ is an α-quasigroup.

3) Clearly $P \subseteq Q_P$, and if $a = h_\tau \, \mathbf{a}$ in $(P, \mathbf{H})$, then $a = \overline{h}_\tau \, \mathbf{a}$ in $(Q_P, \overline{\mathbf{H}})$ Thus, $(P, \mathbf{H})$ is a partial α-subquasigroup of $(Q_P, \overline{\mathbf{H}})$.

Let $u \in Q_P$, and denote by $\bar{u}$ the continued product in the quasigroup $(Q_P, \overline{\mathbf{H}})$ which is obtained from u in such a way that each operator symbol f_τ is replaced by the corresponding operation $\overline{h}_\tau$. By (5.5) we have $\bar{u} = u$. This implies that $(P, \mathbf{H})$ is a generating partial α-subquasigroup of $(Q_P, \overline{\mathbf{H}})$.

4) Let $\lambda : P \to Q'$ be a homomorphism from $(P, \mathbf{H})$ into an α-quasigrou p. $(Q', \mathbf{H}')$. We extend λ to a mapping $\varphi : Q_P \to Q'$ in the following way:

$$a \in P \Rightarrow \varphi(a) = \lambda(a),$$

$$v = f_\gamma \langle u_\beta : \beta \in \alpha \rangle \Rightarrow \varphi(v) = h_\gamma' \ \langle \varphi(u_\beta) : \beta \in \alpha \rangle.$$

It can be easily shown that φ is a homomorphism from $(Q_P, \overline{\mathbf{H}})$ into $(Q', \mathbf{H}')$ Thus, $(Q_P, \overline{\mathbf{H}})$ is freely generated by $(P, \mathbf{H})$. ■

Now we can state the following three corrolaries.

5.2. Every set P freely generates an α-quasigroup $(Q_P, \mathbf{H})$.

Proof. If we put $D_\gamma = \varnothing$ for each $\gamma \in \alpha+1$ we get a partial α-quasigroup $(P, \mathbf{H})$ and then apply **5.1**. ■

5.3. The category of α-quasigroups admits free products.

Proof. Let $\{(P_i, \mathbf{H}^i) | i \in I\}$ be a collection of partial α-quasigroups, such that $\{P_i \mid i \in I\}$ is a collection of disjoint sets. Put $P = \cup P_i$ and define a colection $\langle h_\gamma : \gamma \in \alpha + 1\rangle$ of α-operations on P by:

$$h_\gamma \, a = b \text{ in } P \Leftrightarrow (\exists \, i \in I) \, h^i{}_\gamma \, \mathbf{a} = b \text{ in } P_i .$$

Then $(P, \mathbf{H})$ is a partial α-quasigroup, and the α-quasigroup $(Q_P, \overline{\mathbf{H}})$ freely generated by $(P, \mathbf{H})$ is the free product of the given collection of α-quasigroups. ■

5.4. If α is finite then the word problem in the variety of α-quasiprousp is soluble.

Proof. This is a consequence of **5.1** and the main result of the paper [3].

6. Identities in the varieties of quasiproups. Let X be a well ordered set with type ξ_α, where ξ_α is defined as in the begining of **4**. Elements of X are called *free variables*. If $u \in \Omega_X$, i. e. $\bar{u}$ is a word on X, then (as in the proof of **5.1**) by $\bar{u}$ is denoted the *continued product* in the free quasigroup $(Q_X, \overline{\mathbf{H}})$ which is obtained from u replacing each operator symbol f_β by the corresponding operation $\overline{h}_\beta$. We say that $u = v$ is an *identity* (in the class of α-quasigroups) iff $\bar{u} = \bar{v}$ in the quasigroup Q_X.

6.1. If u and v are different members of Q_X then $u = v$ is not an identity.

Proof. If $u \in Q_X$, then we have $u = \bar{u}$. ■

6.2. Let u, v be two words on X such that v is obtained from u in such a way that some subwords of u of the forms (5.1), (5.2) or (5.3) are replaced by the corresponding subwords u_γ. Then $u = v$ is an identity.

Proof. If w is any word with one of the forms (5.1) — (5.3), then we have $\overline{w} = \bar{u}_\gamma$. ■

6.3. For each Ω-word u there is a unique $v \in Q_X$ such that $u = v$ is an identity.

Proof. By **6.1** we need to show the existence of $v \in Q_X$ with the desired property. This will be shown by induction on the range λ of u. If $\lambda = 0$, then $u \in X \subseteq Q_X$. Assume that $u = f_\gamma \langle u_\beta : \beta \in \alpha\rangle$; then $r(u_\beta) < \lambda$, and therefore there exists an α-sequence $\langle v_\beta : \beta \in Q \rangle \in Q_X^\alpha$ such that $u_\beta = v_\beta$ is an identity for each $\beta \in \alpha$; then $u = v$ is an identity too, where $v = f_\gamma \langle v_\beta : \beta \in \alpha \rangle$; if v does not belong to Q_X then it has one of the forms (5.1) — (5.3) (for, v_β is reduced), and thus there is a $\delta \in \alpha$ such that $u = v_\delta$ is an identity. ∎

6.4 Let α be a finite ordinal. There is an algorithm for answering the question whether an identity $u = v$ holds in the class of α-quasigroups.

Proof. Let $u \in \Omega_X$ and denote by $R(u)$ the word which is obtained from u when the first occurence (from the left on the right, say) of a subword which has one of the forms (5.1) — (5.3) is replaced by u_γ. Denote by $k(u)$ the least positive integer such that $R(R^{k(u)}(u)) = \bar{u} = R^{k(u)}(u)$. Then, $R^{k(u)}(u) \in Q_X$, and $u = R^{k(u)}(u)$ is an identity. Thus $u = v$ is an identity iff $\bar{u} = \bar{v}$. (We note also that **6.4** is a special case of **5.4**.)

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ЗА КВАЗИГРУПИТЕ

(Р е з и м е)

Во работава се воведува поим за α-*квазигрупи*, каде α е (конечен или бесконечен) ординален број. Во првите четири дела се изнесуваат природни обопштувања на соодветните ,,бинарни'' дефиниции и резултати. Во петтиот дел се покажува дека секоја делумна α-квазигрупа може да се смести во α-квазигрупа, што е и главниот резултат на оваа работа. Сметаме дека доказот на овој резултат е поедноставен од познатите докази на соодветниот резултат за бинарните квазигрупи. Неколку резултати во врска со идентитетите во класата α-квазигрупи се докажуваат во последниот дел.

Да споменеме дека во работите [4] и [5] е покажано дека постојат само тривијални (т. е. едноелементни) ω-групи. Природно се наметнува прашањето дали истото важи и за α-групите при бесконечен ординален број α.