SEMIGROUPS IN WHICH EVERY n-SUBSEMIGROUP IS A SUBSEMIGROUP

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1. Introduction. Every subsemigroup Q of a semigroup S is an n-subsemigroup ($Q \subseteq S$, $Q^2 \subseteq Q \Rightarrow Q^{n+1} \subseteq Q$) for every positive integer n. Since the converse is not true, it is of interest to investigate the semigroups which, for some fixed integer $n \geqslant 2$, have the following property:

$$Q \subseteq S, \quad Q^{n+1} \subseteq Q \Rightarrow Q^2 \subseteq Q.$$

Such a semigroup S will be called a *semigroup with the n-property*. This problem is a special case of the problem 6.1 of the paper [4], p. 20.

In this paper we will give some general properties of the semigroups with the n-property and also a structural description of unipotent semigroups with this property, showing previously that every semigroup with the n-property, for $n = 2k$, can be represented as a disjoint union of unipotent semigroups.

2. Some general properties of the semigroups with the n-property. Throughout this section S will denote a semigroup with the n-property. It is obvious that the following lemma holds:

Lemma 1. *Every subsemigroup of a semigroup with the n-property possesses the n-property.* ■

We shall prove that:

Lemma 2. *If S is a semigroup with the n-property, then S is periodic. The index of $\langle a\rangle$, for every $a \in S$, is not greater than* 2 *and the period of $\langle a\rangle$, is relatively prime with n.*

Proof. Let $\langle a\rangle_n$ be the cyclic n-subsemigroup of S generated by a, and let $\langle a\rangle$ be the cyclic subsemigroup of S generated by a. Then

$$\langle a\rangle_n = \{a, a^{n+1}, a^{2n+1}, \ldots\},$$

and, obviously, $\langle a\rangle_n \subset \langle a\rangle$. Since $\langle a\rangle_n$ is a subsemigroup of S too, it has to be $a^2 \in \langle a\rangle_n$; also $a^3 \in \langle a\rangle_n$ etc. and therefore $\langle a\rangle_n = \langle a\rangle$. Since $a^2 \in \langle a\rangle_n$ it follows that

$$a^2 = a^{kn+1} \quad \text{for some } k = 0, 1, 2, \dots,$$

which means that $\langle a\rangle$ is a finite cyclic subsemigroup with index 1 or 2. Let the period of $\langle a\rangle$ be m. By $a^2 = a^{kn+1}$ it follows that $kn \equiv 1 \pmod{m}$, i.e. m and n are relatively prime. ∎

Let us denote by E the set of all idempotents of S. By Lemma 2, for every $x \in S$ there exists a unique idempotent $e \in E$ such that $x^k = e$ for some $k \in N$ ([1], Theorem 1. 9). If we put

$$S_e = \{x \in S \mid x^k = e\}, \qquad e \in E,$$

then by the above argument we come to the following

Lemma 3. $S = \cup\{S_e \mid e \in E\}$, *where the union is disjoint.* ∎

There exists a maximal subgroup H_e of S for every $e \in E$ ([1], Theorem 1.11). Since S is periodic, the subgroup H_e is periodic too and so, if $x \in H_e$, then $x^k = e$ for some $k \in N$. Hence $H_e \subseteq S_e$. Put

$$I_e = S_e \setminus H_e.$$

Some characteristics of the subset I_e of S_e are given in the following

Lemma 4. *If* $x \in S_e$, *then the following statements are equivalent*:

(i) $x \in I_e$,

(ii) *the index of* $\langle x\rangle$ *is* 2.

(iii) $xe = ex = x^{m+1} \neq x$, *where* m *is the period of* $\langle x\rangle$.

Proof. Let (i) holds. By Lemma 2, the index of $\langle x\rangle$ is 1 or 2. If the index is 1, then $\langle x\rangle$ is a subgroup of S and $\langle x\rangle \subseteq S_e$ since $x \in S_e$. Let K_x be the subgroup of $\langle x\rangle$ defined as in Theorem 1.9 of [1]. Since $e \in H_e \cap K_x$ it follows that $K_x \subseteq H_e$ ([1], Theorem 1.11) and $k_x = \langle x\rangle$ (in this case) implies $x \in H_e$ which contradicts (i). Thus the index of $\langle x\rangle$ is 2.

Assume (ii) and let m be the period of $\langle x\rangle$. If $m = 1$, then $x^2 = e$ and then $ex = x^2 x = x^2 = e = xe\ (\neq x)$. If $m \geqslant 2$, then $x^m = e$ and $ex = x^m x = x^{m+1} \neq x$; similarly, $xe = x^{m+1}$. If (iii) holds and if $x \in H_e$, then $x = xe = x^{m+1}$, which is a contradiction. Thus $x \in I_e$. ∎

In general the subsets S_e of a periodic semigroup S are not subsemigroups of S ([2], III. 4. 12). Three sufficient conditions for a periodic semigroup S to have the property: every subset S_e is a subsemigroup of S, are given in [2], III. 4. 5. We shall see later that, in our case, when n is even, all subsets S_e are subsemigroups of S. First we shall prove the following

Lemma 5. *Let S_e, H_e and I_e be defined as above.*

(i) *If $x,y \in S_e$ and at least one of them is in H_e, then $xy \in H_e$.*

(ii) $x S_e^k x \subseteq S_e$ for every $x \in S_e$, $k \in N$.

Proof. If $x, y \in H_e$, then (i) is true since H_e is a subgroup of S. Let, say, $x \in I_e$ and $y \in H_e$ and let the period of $\langle x \rangle$ be m. Then, by Lemma 4, $xy = x(ey) = (xe)y = x^{m+1}y \in H_e$ since $x^{m+1} \in K_x \subseteq H_e$ (the proof of the fact that $K_x \subseteq H_e$ is given in the proof of Lemma 4). Similarly $xy \in H_e$ when $x \in H_e$, $y \in I_e$. This proves (i). Now we shall prove (ii). According to (i), it suffices to consider the case $x, y_i \in I_e$. Let the period of $\langle x \rangle$ be m and let us put $u = xy_1 \ldots y_k x$. By Lemma 3, $u \in S_f$ for some $f \in E$, and then by Lemma 2, $u^2 \in K_u$, where K_u is as in the proof of Lemma 4. Since $K_u \subseteq H_f$, it follows that $u^2 \in H_f$. On the other hand, $u^2 = xy_1 \ldots y_k x^2 y_1 \ldots y_k x$ is in H_e since $x^2 \in H_e$. Thus $H_f \cap H_e \neq \varnothing$ which, because of the maximality of H_e and H_f, implies $H_f = H_e$, and then $e = f$. Therefore $u \in S_e$. ∎

Let n be an even number and let $x, y \in S_e$. Let us denote by $\langle x, y \rangle_n$ the n-subsemigroup of S generated by $\{x, y\}$. Any element $w \in \langle x, y \rangle_n$ can be represented as a product of x's and y's, all of them together contained in w $kn+1$ times. If two neighbouring factors in w are equal, then by (i) of Lemma 5, $w \in H_e$ since $x^2, y^2 \in H_e$. If this is not the case, then w will be of the form u, defined as in Lemma 5. aBy (ii) of this lemma, $w \in S_e$, which means that all elements of $\langle x, y \rangle_n$ ire in S_e. Since $\langle x, y \rangle_n$ is also a subsemigroup of S containing x and y, t follows that $xy \in \langle x, y \rangle_n$ and by the above argument, $xy \in S_e$.

Let Q be a unipotent subsemigroup of S with e as its idempotent and let $x \in Q$. Then $\langle x \rangle \subseteq Q$; by Lemma 2, $\langle x \rangle$ has an idempotent which equals e, since Q is unipotent. Therefore $x \in S_e$, i. e. $Q \subseteq S_e$. Hence S_e is the maximal unipotent subsemigroup of S with e as its idempotent.

By this and Lemmas 2 and 3, we obtain the following

Theorem 1. *Let S be a semigroup with the property: every n-subsemigroup of S, for some $n = 2k$, $k \in N$ fixed, is a subsemigroup of S. Then S is periodic and:*

(i) *the index of $\langle a \rangle$ is not greater than 2, and the period of $\langle a \rangle$ is relatively prime with n for every $a \in S$;*

(ii) $S = \cup \{S_e \mid e \in E\}$, *where the union is disjoint;*

(iii) *S_e is the maximal unipotent subsemigroup of S with e as its idempotent, H_e is a (unique maximal) subgroup of S_e which is also an (two-sided) ideal in S_e.*

Here E, S_e and H_e are defined as before Lemma 3 and after it. ∎

It asserts in this theorem that every S_e, $e \in E$, is a subsemigroup of S when n is an even number. But if n is not even, then S_e may not be a subsemigroup, which is seen by the following

Example. Let $S = \{O, A, B, C, D\}$, where

	O	A	B	C	D
O	O	O	O	O	O
A	O	A	B	O	O
B	O	O	O	A	B
C	O	C	D	O	O
D	O	O	O	C	D

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and the operation in S is the matrix multiplication (or the operation in S is defined by the given Cayley table). The idempotents in S are O, A and D, and the corresponding unipotent subsets of S are:

$$S_o = \{O, B, C\}, \quad S_A = \{A\}, \quad S_D = \{D\}.$$

S_o is not a subsemigroup of S since $BC = A \in S_A$. All the subsemigroups of S are: $\{O\}$, $\{A\}$, $\{D\}$; $\{O, A\}$, $\{O, B\}$, $\{O, C\}$, $\{O, D\}$; $\{O, A, B\}$, $\{O, A, C\}$, $\{O, A, D\}$, $\{O, B, D\}$, $\{O, C, D\}$; $\{O, A, B, D\}$, $\{O, A, C, D\}$; S. The other subsets of S, except the empty set, which are not subsemigroups of S, are: $\{B\}$, $\{C\}$; $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$; $\{O, B, C\}$, $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, $\{B, C, D\}$; $\{O, A, B, C\}$, $\{O, B, C, D\}$, $\{A, B, C, D\}$. None of these subsets which contains at least one of the elements B, C but not O, is an n-subsemigroup of S since $B^{n+1} = O = C^{n+1}$ for every $n \geqslant 2$. The subset $\{A, D\}$ is not an n-subsemigroup of S since, for examples $AD^n = AD = O$. It remains to check whether or not some of the subset, $P = \{O, B, C\}$, $Q = \{O, A, B, C\}$, $R = \{O, B, C, D\}$ is an n-subsemigroup for some odd n. Let us take $n = 3$. Then P and R are not 3-subsemigroups since $BCBC = AA = A$, and also Q is not 3-subsemigroup of S since $CBCB = DD = D$. Hence S is a semigroup with the 3-property in which S_o is not a subsemigroup.

3. The structure of the unipotent semigroups with the n-property. Throughout this section S will denote a unipotent semigroup with the n-property, for an arbitrary fixed $n \geqslant 2$, in which the idempotent will be denoted by e. By H will be denoted the maximal subgroup of S, and by I the set of all elements of S which are not in H. We shall give a structural description of S a little later. First we shall prove the following

Lemma 6. *For any* $x, y \in S$, $\quad xy \in H$.

Proof. If at least one of the elements x, y belongs to H, then the assertion is true by (i) of Lemma 5. Let $x, y \in I$ and let $Q = H \cup \{x, y\}$ when $n > 2$, and $Q = H \cup \{x, y, xyx, yxy\}$ when $n = 2$. We will show that Q is an n-subsemigroup of S. Let us consider first the case $n > 2$. Let $q_j \in Q$, $j = 0, 1, \ldots, n$, and let $u = q_0 q_1 \ldots q_n$. If $q_j \in H$ for some $j = 0, 1 \ldots, n$, then by Lemma 5, $u \in H$. Let $q_j \notin H$ for all $j = 0, 1, \ldots, n$. Then u is a product of x's and y's, such that one of the following cases holds: either in u one of x^2, y^2 will occur as a factor, and then $u \in H$ since $x^2, y^2 \in H$, or u will be one of the products $xyxy \ldots xy$, $xyxy \ldots xyx$, $yxyx \ldots yx$, $yxyx \ldots yxy$ depending on whether n is even or odd. In the first case $u = xyxy \ldots xy = (xy)^2 x \ldots xy \in H$ since $(xy)^2 \in H$; similarly we infer that $u \in H$ in the other cases. The proof is similar for $n = 2$; namely, if x and y occur as factors in u more than three times, then we can repeat the above argument; if they occur as factors only three times, then either $u \in H$ or u will be equal to one of the products xyx, yxy and so in any case $u \in Q$.

Now, having in mind that Q is a subsemigroup, we get that $xy \in Q$ and so in the case when $n > 2$, for xy one of the following three cases is possible: $xy \in H$, $xy = x$, $xy = y$. If $xy = x$, then $xy = (xy)\, y = xy^2 \in H$ since $y^2 \in H$. Similarly, from $xy = y$ it follows that $xy \in H$; thus, in any case, $xy \in H$. Two more cases are possible for $n = 2$: $xy = xyx$ or $xy = yxy$; if $xy = xyx$, then $xy = xyx = (xyx)\, x = xy \cdot x^2 \in H$ and similarly, if $xy = yxy$, then again $xy \in H$. ■

A sight into the structure of the unipotent semigroups with the n-property is obtainable by the Theorem of [3] since these semigroups are particular cases of the semigroups studied in [3]. But, as the semigroups studied here are richer in properties than those in [3], the description of their structure is simpler.

First we shall prove:

Lemma 7. *A group H has the n-property if and only if H is periodic such that the order of each of its elements is relatively prime with n.*

Proof. According to Lemma 2 we have to prove only the „if" part. Let Q be an n-subsemigroup of the group H where $n \geqslant 2$ is fixed. Let $x \in H$ and denote by m the order of $\langle x \rangle$. Then $sm = rn + 1$, where s and r are integers and $r \in \mathbf{N}$, and then $e = x^{sm} = x^{rn+1} \in Q$, where e is the identity in H. Now if $u, v \in Q$, then $uv = uve^{n-1} \in Q$, i. e. Q is a subsemigroup of H; moreover, it is a subgroup of H. ■

Let H be a periodic group such that the order of each of its elements is relatively prime with n, and let I be a set disjoint with H. Let $f\colon H \cup I \to H$ be a mapping such that $f | H = 1_H$. If we define an operation „$\bigcirc$" on $S = H \cup I$ by

$$x \bigcirc y = f(x) f(y),$$

then:

Lemma 8. *$S(\bigcirc)$ is a unipotent semigroup with the n-property in which H is the maximal subgroup.*

Proof. Obviously $S(\bigcirc)$ is a semigroup. Let Q be an n-subsemigroup of S and $x \in Q$. Then the order m of $f(x) \in H$ is relatively prime with n and, as in Lemma 7, this implies that the identity $e \in H$ belongs to Q. Again as in Lemma 7, we get that Q is a subsemigroup of S. If z is an idempotent in S, then by the definition of „$\bigcirc$", $z = z^2 \in H$ and z is an idempotent in H, hence $z = e$. Let G be a subgroup of S. Since S is unipotent, the identity e of H is the identity in G too. If $w \in G$, then $w = w \bigcirc e \in H$, again by the definition of „$\bigcirc$", and thus $G \subseteq H$, which proves the maximality of H. ∎

Let us denote by (I, H, f) the semigroup $S(\bigcirc)$ of Lemma 8.

Theorem 2. *A unipotent semigroup S has the n-property if and only if $S = (I, H, f)$ where I, H and f are uniquely determined by S.*

Proof. Let H be the maximal subgroup of S and $I = S \setminus H$. If e is the idempotent of S and if we put $f(x) = xe\ (= ex)$, then $f\colon S \to H$ is a mapping such that $xy = f(x) f(y)$ and $f | H = 1_H$, so $S = (I,\ H,\ f)$.

Conversely, if $S = (I, H, g)$, then

$$g(x) = g(x)\, e = g(x)\, g(e) = x \bigcirc e = f(x),$$

i.e. $g = f$. The proof is completed by Lemma 8. ∎

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РЕЗИМЕ

ПОЛУГРУПИ КАЈ КОИ СЕКОЈА n-ПОТПОЛУГРУПА Е ПОТПОЛУГРУПА

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За полугрупата S велиме дека го има *n-својството* ако секоја нејзина n-потполугрупа, $n \in \mathbf{N}$ фиксен, е потполугрупа. Во работава се изнесуваат неколку: општи особини за полугрупите со n-својството и се докажуваат следниве теореми,

Теорема 1. *Нека S е полугрупа што го има n-својството, каде што $n=2k$ $k \in \mathbf{N}$. Тогаш S е периодична и:*

(i) *Индексот на секоја циклична потполугрупа $\langle a \rangle$ не е поголем од 2, а нејзиниот период е заемно прост со n.*

(iii) *Ако E е множеството од сите идемпотенти од S и $S_e = \{x \in S \mid x^k = e, k \in \mathbf{N}\}$, $e \in E$, тогаш $S = \cup \{S_e \mid e \in E\}$, при што унијата е дисјунктна.*

(ii) *S_e е максималната унипотентна потполугрупа од S со е како нејзин идемпотент* (т.е. *секоја потполугрупа од S што го содржи е како единствен идемпотент е потполугрупа од S_e*).

(iv) *Во секоја полугрупа S_e постои единствена максимална подгрупа H_e, која е и (двостран) идеал во S_e.*

Забелешка. Ако n е непарен број, тогаш S_e не мора да биде потполугрупа од S што е покажано со пример.

Ако H е периодична група таква што редот на секој нејзин елемент е заемно прост со n, а I е множество дисјунктно со H и ако $f: H \cup I \to H$ е такво пресликување што $f \mid H = 1_H$, тогаш со

$$x \bigcirc y = f(x) f(y)$$

е дефинирана асоцијативна операција на $S = H \cup I$. Вака добиената полугрупа ќе ја означиме со (I, H, f).

Теорема 2. *Една унипотентна полугрупа S го има n-својството ако и само ако $S = (I, H, f)$, каде што I, H и f се еднозначно определени со S.*

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