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ON SOME PRIMITIVE CLASSES OF UNIVERSAL ALGEBRAS

The purpose of this note is to characterise some primitive classes of universal algebras with the class of semigroups.

THEOREM 1. *Let $Q(*)$ be a universal algebra with an $(n+1)$-ary operation $*$. There exist a semigroup S and a fixed element a of S such that $Q \subseteq S$ and*

(1) $$* x_0 x_1 \ldots x_n = x_0 a x_1 \ldots x_n$$

for all $x_\nu \in Q$, if and only if the following identity equation is satisfied in $Q(*)$

(2) $$* * x_0 x_1 \ldots x_{2n} = * x_0 \ldots x_{n-1} * x_n \ldots x_{2n}.$$

PROOF. At first, it can be easily seen that if a is an element of a semigroup S, and if the operation $*$ is defined by (1), then the identity (2) holds in $S(*)$.

Suppose that $Q(*)$ is a universal algebra satisfying the identity (2), and denote by F the semigroup which is freely generated by the set $Q \cup \{a\}$, where a is a symbol which does not belong to Q. By putting $*_a x_0 x_1 \ldots x_n = x_0 a x_1 \ldots x_n$ $(x_\nu \in F)$, we obtain an algebra $F(*_a)$ in which the identity (2) is satisfied.

If $i_1, i_2, \ldots, i_k$ is a sequence of non-negative integers such that $0 \leqslant i_{\nu-1} \leqslant i_\nu \leqslant \nu n$, then the following $(kn+1)$-ary operation $* i_1 i_2 \ldots i_k$ can be defined (in an arbitary algebra with a finitary operation $*$)

(3) $$* i_1 \ldots i_k x_0 \ldots x_{(k+1)n} = * x_0 \ldots * x_{i_1} \ldots * x_{i_2} \ldots * x_{i_k} \ldots x_{(k+1)n}.$$

Let $u \in F$ and

(4) $$u = *_a i_1 \ldots i_k a_0 a_1 \ldots a_{(k+1)n},$$

for some $a_\lambda \in Q$ and non-negative integers i_ν. We may assume that $0 < i_{\nu-1} < i_\nu \leqslant \nu n$. (For, if this is not satisfied, every subproduct $*_a *_a z_0 \ldots z_{2n}$ can be replaced by $*_0 z_0 \ldots z_{n-1} * z_n \ldots z_{2n}$, and after a finite number of such transformations the product u will obtain the following form $u = *_a i_1' i_2' \ldots i_k' \boldsymbol{a}$ where $0 < i_{\nu-1}' < i_\nu' \leqslant \nu n$ and $\boldsymbol{a} = (a_0, a_1, \ldots a_{(k+1)n})$).

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Suppose that the following equation is also true

$$(4') \qquad u = *_a j_1 j_2 \ldots j_r \boldsymbol{b},$$

where $b_\lambda \in Q$ and $0 < j_{\nu-1} < j_\nu \leqslant \nu n$. From (4) and (4′) follows that the following equation is satisfied in the free semigroup F

$$(4'') \quad a_0 a \ldots a a_{i_1} \ldots a a_{i_k} \ldots a_{(k+1)n} = b_0 a \ldots a b_{j_1} \ldots a b_{j_r} \ldots b_{(r+1)n}.$$

And, this equation implies $r=k$, $i_\nu = j_\nu$ and $a_\lambda = b_\lambda$. Thus, the product $u = *_a i_1 \ldots i_k \boldsymbol{a}$ is uniquely determined by u, and it is said to be the canonical form of u.

If $x = c_1 \ldots c_m \in F$ and $c_i = * \; b_0 b_1 \ldots b_n$ in $Q(*)$, then we write

$$(5) \qquad x \vdash y = c_1 \ldots c_{i-1} b_0 a b_1 \ldots b_n c_{i+1} \ldots c_m \text{ and } y \dashv x.$$

Let $b \in Q$ and $b \vdash u_1 \vdash u_2 \vdash \cdots \vdash u_k \vdash u$. Clearly, then there exist a sequence $\boldsymbol{a} = (a_0, a_1, \ldots, a_{(k+1)n})$ — $a_\lambda \in Q$ — and non-negative integers $i_1, \ldots, i_k$ — $0 < i_{\nu-1} < i_\nu \leqslant \nu n$ — such that

$$(6) \qquad u = *_a i_1 i_2 \ldots i_k \boldsymbol{a} \text{ in } F \text{ and } b = * \; i_1 i_2 \ldots i_k \boldsymbol{a} \text{ in } Q(*).$$

And conversely, (6) implies $b \vdash u_1 \vdash \cdots \vdash u_k \vdash u$ for some $u_\nu \in F$.

Suppose that $v = c_1 \ldots c_t \vdash u$. Then we have

$$(6') \qquad u = c_1 \ldots c_{s-1} e_0 a e_1 \ldots e_n c_{s+1} \ldots c_t$$

where $* \; e_0 e_1 \ldots e_n = c_s$ in $Q(*)$. By (6) and (6′), there exist p and r $(1 \leqslant p < k)$ such that $i_p = r$, $a_{r+\nu} = e_\nu$ and $i_{p+1} - i_p \geqslant n$. If $i_{p+1} - i_p > n$, then we have

$$(6'') \qquad \begin{aligned} u &= *_a i_1 \ldots i_{p-1} j_p \ldots j_{k-1} \boldsymbol{a}_1 \; *_a \underline{a}_r \ldots a_{r+n} \boldsymbol{a}_2 \\ v &= *_a i_1 \ldots i_{p-1} j_p \ldots j_{k-1} \boldsymbol{a}_1 c_s \boldsymbol{a}_2, \end{aligned}$$

and

$$(6''') \qquad \begin{aligned} b &= * \; i_1 \ldots i_{p-1} j_p \ldots j_{k-1} \boldsymbol{a}_1 * a_r \ldots a_{r+n} \boldsymbol{a}_2 \\ &= * \; i_1 \ldots i_{p-1} j_p \ldots j_{k-1} \boldsymbol{a}_1 c_s \boldsymbol{a}_2, \end{aligned}$$

where $(\underline{\boldsymbol{a}}_1, d, \underline{\boldsymbol{a}}_2) = (a_1, \ldots, a_{r-1}, d, a_{r+n+1}, \ldots, a_{(k+1)n})$ and $\boldsymbol{i}_{p+\nu+1} - n = j_{p+\nu}$. In the case $i_{p+1} - i_p = n$, it can be easily seen that the equation (6″) and (6‴) are satisfied with $i_p = j_p$, $i_{p+\nu+2} - n = j_{p+\nu+1}$.

Thus, we have proved that $b \in Q$ and $b \vdash u_1 \vdash \cdots \vdash u_k \vdash u \dashv v$ imply

$$(7) \qquad v = *_a j_1 \ldots j_{k-1} \boldsymbol{d} \text{ in } F \text{ and } b = * \; j_1 \ldots j_{k-1} \boldsymbol{d} \text{ in } Q(*),$$

for some $d_\lambda \in Q$ and $0 < j_{\nu-1} < j_\nu \leqslant \nu n$. By a finite number of applications of this result, we obtain the following

LEMMA. *If* $b, c \in Q$ *and if* $b \vdash\!\dashv_1 w_1 \vdash\!\dashv_2 w_2 \vdash\!\dashv_3 \ldots \vdash\!\dashv_k w_k \vdash\!\dashv_{k+1} c$ *for some* $\vdash\!\dashv_\nu \in \{\vdash, \dashv\}$ *and* $w_\nu \in F$, *then* $b = c$.

Denote by ρ the minimal transitive and reflexive relation in F such that $u \vdash v$ or $u \dashv v \Rightarrow u \rho v$; that is

$$(8) \qquad u \rho v \Leftrightarrow u = v \text{ or } u \vdash\!\dashv_1 w_1 \vdash\!\dashv_2 \ldots \vdash\!\dashv_k w_k \vdash\!\dashv_{k+1} v$$

for some $\vdash\!\dashv_\nu \in \{\vdash, \dashv\}$ and $\omega_\nu \in F$. Clearly, ρ is a congruence in the semigroup F, and denote by S the corresponding factor-semigroup F/ρ. If $x \in F$, then

$x\rho=\{y;\ y\in F,\ x\rho y\}$ is an element of S, and it may be assumed that $a\in S$, because $a\rho=\{a\}$. If $b=*\ b_0 b_1 \ldots b_n$ in $Q(\ *)$, then we have

$$b\rho\, b_0 a b_1 \ldots b_n = *_a b_0 b_1 \ldots b_n,$$

and this implies that the mapping $b\to b\rho$ is a homomorphism of $Q(\ *\ )$ onto $Q_\rho(\ *_a)$, where $Q_\rho=\{b\rho;\ b\in Q\}$. By Lemma, we have $b,c\in Q,\ b\rho c\Rightarrow b=c$, and therefore the homomorphism $b\to b\rho$ is an isomorphism. This completes the proof of Theorem.

The following three theorems can be proved in a quite similar manner. (Theorem 2 is a generalization of Theorem 1)

THEOREM 2. *Let $Q(\Phi)$ be a universal algebra with a system Φ of finitary operations, at most one of which is unary. There exit a semigroup S and a collection $\{a_*;\ *\in\Phi\}$ of elements of S such that $Q\subseteq S$ and $*\,x_0x_1\ldots x_{n_*}=x_0a_*x_1\ldots x_{n_*}$ for all $x_\nu\in Q$, $*\in\Phi$, if and only if the following identity equation is satisfied in $Q(\Phi)$:*

$$(9)\qquad *\ \square\, x_0 x_1 \ldots x_{n_*+n_\square} = \square\, x_0 \ldots x_{n_\square-1} * x_{n_\square} \ldots x_{n_*+n_\square},$$

for all $*,\square\in\Phi$.

And, there exists a commutative semigroup S with the above mentioned properties if and only if the all operations of Φ are commutative; that is — for every operation $*\in\Phi$ and permutation ξ of $0, 1, \ldots, n_*$ the following identity is satisfied in $Q(\Phi)$: $*\ x_0x_1\ldots x_{n_*}=*\ x_{0\xi}x_{1\xi}\ldots x_{n_*\xi}$. (The assumption that there does not exist more than one unary operation is not necessary in the second part of Theorem)

THEOREM 3. *Let $Q(\Phi)$ be a universal algebra with a system of finitary operation Φ, such that two different operations belonging to Φ have different lengths; we also assume that the identity unary operation (on Q) belongs to Φ. Then the algebra $Q(\Phi)$ can be embedded in a semigroup $S_{(\cdot)}$ such that $*\ x_0x_1\ldots x_{n_*}=x_0.x_1\ldots x_{n_*}$ (for all $*\in\Phi$ and $x_\nu\in Q$), if and only if the following two statements are satisfied:*

1°: *If $*_1,\ldots,*_s,\square_1,\ldots,\square_t\in\Phi$ and $n=n_{*1}+\cdots+n_{*_s}=n_{\square 1}+\cdots+n_{\square_t}$ then the following identity is satisfied*

$$(10)\qquad *_1\ldots *_s x_0\ldots x_n=\square_1\ldots\square_t x_0\ldots x_n.$$

2°: If $*,\ \square\in\Phi$ then the following identity is satisfied in $Q(\Phi)$

$$(11)\qquad *\ x_0\ldots x_{r-1}\,\square\, x_r\ldots x_{n_*+n_\square}=\square\, x_0\ldots x_{s-1} * x_s\ldots x_{n_*+n_\square}$$

for every pair $r,s: 0\leqslant r\leqslant n_*$ and $0\leqslant s\leqslant n_\square$.

If all the operations of Φ are commutative, then $Q(\Phi)$ can be embedded in a commutative semigroup. (This Theorem is a generalization of the main result of the paper [5]).

THEOREM 4. *If $Q(\Phi)$ is a universal algebra with finitary operations, then there exist a semigroup $S_{(\cdot)}$ and a collection $\{a^*;\ *\in\Phi\}$ of elements of S such that $Q\subseteq S$, and the equation $*\ x_0x_1\ldots x_{n_*}=a^*x_0x_1\ldots x_{n_*}$ is satisfied for all $*\in\Phi$ and $x_\nu\in Q$.*

Theorem 4 is proved by Ребане in his (unpublished) paper [2], and we are informed by him that the same result is also proved by Cohn in his monography Universal Algebra (New York 1965). A proof of the second part of Theorem 2 is also given by Ребане in [3].

EXAMPLE. Let $Q(\Phi)$ be a universal algebra with finitary operations, such that

$$* \square x_{00} x_{01} \ldots x_{0n} \square x_{10} x_{11} \ldots x_{1n} \square \ldots \square x_{m0} x_{m1} \ldots x_{mn} = \square * x_{00} x_{10} \ldots x_{m0} * x_{01} x_{11} \ldots x_{m1} * \ldots * x_{0n} x_{1n} \ldots x_{mn} \tag{12}$$

is an identity for every pair of operations $*, \square \in \Phi$. Then $Q(\Phi)$ is said to be an entropic algebra [1]. Let Φ' be a system of polynomials (without constant coefficients) in the algebra $Q(\Phi)$. Then the algebra $Q(\Phi')$ is also entropic (see [4] p. 32).

PROBLEM. Is it true that every entropic algebra can be imbedded in an entropic algebra with binary operations? And is it true that every entropic algebra with a countable system of operations can be imbedded in an entropic groupoid?

REFERENCES

[1] T. Evans, *Abstract mean values*, Duke Math. Journ. 30 (1963) 331—347.

[2] Ю. К. Ребане, *Об одном представлении произвольных универсальных алгебр в полугруппах* (unpublished).

[3] Ю. К. Ребане, *О представлении универсальных алгебр в коммутативных полугруппах*, Сиб. Мат. Журн. 7 (1966) 878—885.

[4] Г. Чупона, *За финитарните операции*, Год. Зб. Прир. мат. фак. Скопје 12 (1959) 7—49.

[5] Г. Чупона, *За асоцијативните конгруенции*, Билтен Друшт. мат. физ. СР Макед. 13 (1962) 5—12.