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ON COMPLETELY SIMPLE SEMIGROUPS

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A semigroup S is called completely simple without zero[1], if it contains a minimal left ideal and a minimal right one, and if it does not contain proper two-sided ideals.

Let: (*i*) G be a group; (*ii*) L and R be two non-empty sets, (*iii*) λ be a mapping of $R \times L$ in G, (*iv*) $S = G \times L \times R$, and (*v*) a product of two elements of S be defined by

$$(g_1 ; l_1, r_1)(g_2 ; l_2, r_2) = (g_1 \lambda (r_1, l_2) g_2 ; l_1, r_2) \quad (g_i \in G, l_i \in L, r_i \in R).$$

Then $S = S(G; L, R; \lambda)$ is a completely simple semigroup. If $A = L \times R$ and $G_\alpha = \{(x ; \alpha), x \in G\}$, then $F = \{G_\delta ; \delta \in A\}$ is the collection of all maximal subgroups of S, and the following two equations are also true:

$$S = \bigcup_{\delta \in A} G_\delta, \tag{1}$$

$$G_\delta S G_\delta = G_\delta, \text{ for every } \delta \in A. \tag{2}$$

It is well known (see, for example [2] p. 500 or [3] p. 291) that each completely simple semigroup (without zero) is isomorphic with a semigroup $S(G; L, R; \lambda)$. Therefore, if $F = \{G_\delta ; \delta \in A\}$ is the collection of all maximal subgroups of the completey simple semigroup S, then the statements (1) and (2) are valid.

The purpose of this paper is to prove the following results.*

T h e o r e m. *Let S be a semigroup such that there exists a collection $F = \{G_\delta ; \delta \in A\}$ of subgroups of S which satisfy* (1) *and* (2). *Then the semigroup S is completely simple, and F is the collection of all maximal subgroups of S.*

C o r o l l a r y 1. *A completely simple semigroup S is isomorphic with the direct product of a group G and an anticommutative semigroup*[2] *A if and only if the set B of all idempotents of S is a subsemigroup of S; then the semigroups A and B are isomorphic.*

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[1] Henceforth, the phrase »without zero« will be omitted.

* *Note added in proof.* Most of results of this paper are to be found in the book A. H. C l i f f o r d and G. B. P r e s t o n, The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys № 7, Amer. Math. Soc. 1961. (Ex. 14 and 15, p. 84 § 2.7, Ex. 2 (b), p. 97 § 3.2), which was not available to the author at the time he submitted the paper for publication.

[2] A semigroup A is anticommutative if $\alpha\beta = \beta\alpha$ implies $\alpha = \beta$, $\alpha, \beta \in A$.

C o r o l l a r y 2. *If the semigroup S is a union of subgroups each of which is a left ideal of S, then there exist a group G and a set A such that S is isomorphic with the semigroup $G \times A$, where* $(x, \alpha)(x, \beta) = (x, \beta)$.

First, in Sections **1** and **2**, we prove some more general lemmas, and then, in Section **3**, we prove the Theorem and the Corollaries.

1. Let G_α, G_β and G_γ be subgroups of the semigroup S such that the statements $G_\alpha G_\beta \cap G_\gamma \neq 0$ and (2) hold, for $\delta = \alpha, \beta, \gamma$.

L e m m a 1.1. *Let* $\varepsilon, \delta = \alpha, \beta, \gamma$. *If* $G_\varepsilon \cap G_\delta \neq 0$, *then* $G_\varepsilon = G_\delta$.

P r o o f. Let $x \leqslant G_\varepsilon \cap G_\delta$. Then $G_\varepsilon = x G_\varepsilon x \subseteq G_\delta S G_\delta = G_\delta$. Analogously, $G_\delta \subseteq G_\varepsilon$.

L e m m a 1.2. *If* $a_\alpha \leqslant G_\alpha, a_\beta \leqslant G_\beta$ *and* $a_\alpha a_\beta \leqslant G_\gamma$, *then* $G_\gamma \subseteq \subseteq G_\alpha a_\beta \cap a_\alpha G_\beta$.

P r o o f. We have $G_\gamma = a_\alpha a_\beta G_\gamma a_\alpha a_\beta \subseteq a_\alpha a_\beta S a_\beta \subseteq a_\alpha G_\beta$. Analogously, $G_\gamma \subseteq G_\alpha a_\beta$.

L e m m a 1.3. $G_\alpha = x_\gamma G_\alpha = G_\gamma x_\alpha$[3].

P r o o f. Let a_α and a_β be determined as in Lemma 1.2, i. e. $a_\alpha a_\beta \leqslant G_\gamma$, and let x_γ be an arbitrary element of G_γ. By 1.2, we have $x_\gamma = a_\alpha b_\beta$, for some $b_\beta \leqslant G_\beta$, whence (*i*) $x_\gamma a_\alpha = a_\alpha b_\beta a_\alpha \leqslant G_\alpha$. From (*i*) (by 1.2) it follows (*ii*) $G_\alpha \subseteq G_\gamma a_\alpha \cap x_\gamma G_\alpha$. We have also (*iii*) $G_\gamma a_\alpha \subseteq a_\alpha G_\beta a_\alpha \subseteq G_\alpha$, and (*iv*) $x_\gamma G_\alpha = a_\alpha b_\beta G_\alpha \subseteq G_\alpha$. From (*ii*), (*iii*) and (*iv*), we get (*v*) $G_\alpha = G_\gamma a_\alpha = x_\gamma G_\alpha$, whence (*vi*) $G_\alpha = G_\alpha a_\alpha^{-1} x_\alpha = = G_\gamma a_\alpha a_\alpha^{-1} x_\alpha = G_\gamma x_\alpha$. This completes the proof of the lemma.

In a similar way, the following can be proved.

L e m m a 1.4. $G_\beta = G_\beta x_\gamma = x_\beta G_\gamma$.

A consequence of Lemmas 1.2, 1.3, *and* 1.4 is the following

L e m m a 1.5. $G_\gamma = G_\alpha x_\gamma = x_\alpha G_\gamma = G_\gamma x_\beta = x_\gamma G_\beta$.

The following lemma is a generalization of Lemma 1.2.

L e m m a 1.6. $G_\gamma \subseteq x_\alpha G_\beta \cap G_\alpha x_\beta$.

P r o o f. Let x_α be an arbitrary element of G_α. By Lemma 1.5, $G_\gamma = x_\alpha a_\alpha^{-1} G_\gamma$, whence (by 1.2) $G_\gamma = x_\alpha a_\alpha^{-1} G_\gamma \subseteq x_\alpha a_\alpha^{-1} a_\alpha G_\beta = = x_\alpha G_\beta$. Similarly, $G_\gamma \subseteq G_\alpha x_\beta$.

L e m m a 1.7. *If e_δ is the neutral element of G_δ then*

$$e_\alpha = e_\gamma e_\alpha, \quad e_\beta = e_\beta e_\gamma, \quad e_\gamma = e_\alpha e_\gamma = e_\gamma e_\beta .$$

P r o o f. By 1.3, $e_\gamma e_\alpha \leqslant G_\alpha$. If x_α is an arbitrary element of G_α, then (by 1.3) there is an element b_γ of G_γ so that $x_\alpha = b_\gamma e_\alpha$. Hence, $x_\alpha = b_\gamma e_\alpha = b_\gamma e_\gamma e_\alpha = b_\gamma e_\alpha e_\gamma e_\alpha = x_\alpha e_\gamma e_\alpha$, i. e. $e_\gamma e_\alpha = e_\alpha$. Similarly, $e_\beta = e_\beta e_\gamma$. It is clear that $e_\gamma e_\alpha = e_\alpha$ and $e_\beta e_\gamma = e_\beta$ (according to the above lemmas) imply $e_\gamma = e_\alpha e_\gamma = e_\gamma e_\beta$.

[3] x_δ is an arbitrary element of G_δ.

Lemma 1.8. *The subgroups G_α, G_β and G_γ are isomorphic.*

Proof. Let $x_\alpha \varphi_{\alpha\gamma} = x_\alpha e_\gamma$ and $x_\gamma \varphi_{\gamma\alpha} = x_\gamma e_\alpha$. By 1.7, we have $x_\alpha \varphi_{\alpha\gamma} \varphi_{\gamma\alpha} = x_\alpha$ and $x_\gamma \varphi_{\gamma\alpha} \varphi_{\alpha\gamma} = x_\gamma$, i. e. $\varphi_{\alpha\gamma}$ is an one-to-one mapping G_α onto G_γ and $\varphi_{\gamma\alpha} = \varphi_{\alpha\gamma}{}^{-1}$. We have also

$$(x_\alpha y_\alpha) \varphi_{\alpha\gamma} = x_\alpha y_\alpha e_\gamma = x_\alpha e_\gamma y_\alpha e_\gamma = x_\alpha \varphi_{\alpha\gamma} \cdot y_\alpha \varphi_{\alpha\gamma},$$

i. e. $\varphi_{\alpha\gamma}$ is an isomorphism of G_α on G_γ. If we put $x_\beta \psi_{\beta\gamma} = e_\gamma x_\beta$ and $x_\gamma \psi_{\gamma\beta} = e_\beta x_\gamma$; in a similar way it can be shown that $\psi_{\beta\gamma}$ is an isomorphism of G_β on G_γ and $\psi_{\gamma\beta} = \psi_{\beta\gamma}{}^{-1}$.

2. Let $F = \{G_\delta;\ \delta \in A\}$ be a system of subgroups of the semigroup S, such that the statement (2) holds.

From the Lemmas 1.6 and 1.8 it follows:

Lemma 2.1. *Let $\alpha, \beta \in A$ and let $\gamma \in A' \Leftrightarrow G_\alpha G_\beta \cap G_\gamma \neq 0$. Then $\bigcup_{\gamma \in A'} G_\gamma \subseteq x_\alpha G_\beta \cap G_\alpha x_\beta$. If $A' \neq 0$, then all groups of the system $\{G_\delta; \delta \in A' \cup \{\alpha, \beta\}\}$ are isomorphic.*

Lemma 2.2. *If $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A} G_\delta$, then there is a $G_\gamma \in F$ such that $G_\alpha G_\beta = x_\alpha G_\beta = G_\alpha x_\beta = G_\gamma$, for every $x_\alpha \in G_\alpha, x_\beta \in G_\beta$.*

Proof. Let $A' (\subseteq A)$ be defined as in Lemma 2.1. Then $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A} G_\delta$ implies $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A'} G_\delta$, whence (by 1.6) we obtain

$$(i) \quad \bigcup_{\delta \in A'} G_\delta = x_\alpha G_\beta = G_\alpha x_\beta = G_\alpha G_\beta.$$

Let G_γ and G_ε be two members of the system $\{G_\delta; \delta \in A'\}$, and let $x_\alpha, x_\beta, x_\gamma$ and x_ε be arbitrary elements of $G_\alpha, G_\beta, G_\gamma$ and G_ε respectively. Then, by (*i*), there are elements $y_\alpha, z_\alpha \in G_\alpha$ and $y_\beta, z_\beta \in G_\beta$ such that (*ii*) $x_\gamma = y_\alpha x_\beta = x_\alpha y_\beta$, and (*iii*) $x_\varepsilon = z_\alpha x_\beta = x_\alpha z_\beta$. From (*ii*) and (*iii*) it follows:

$$(iv) \quad x_\gamma x_\varepsilon = x_\gamma z_\alpha x_\beta \in x_\gamma G_\alpha x_\beta = x_\gamma G_\alpha y_\alpha x_\beta = x_\gamma G_\alpha x_\gamma = G_\gamma$$

and

$$(v) \quad x_\gamma x_\varepsilon = x_\alpha y_\beta x_\varepsilon \in x_\alpha G_\beta x_\varepsilon = x_\alpha z_\beta G_\beta x_\varepsilon = x_\varepsilon G_\beta x_\varepsilon = G_\varepsilon,$$

i. e.

$$(vi) \quad x_\gamma x_\varepsilon \in G_\gamma \cap G_\varepsilon.$$

From (*vi*), by 1.1, it follows $G_\varepsilon = G_\gamma$. Therefore, we have $A' = \{\gamma\}$ and this proves our lemma.

3. Proof of the Theorem. Let $F = \{G_\delta; \delta \in A\}$ be a collection of (different) subgroups of the semigroup S such that the statements (1) and (2) hold.

If we put $\alpha\beta = \gamma \Leftrightarrow G_\alpha G_\beta = G_\gamma$, then (by 2.2) A becomes a semigroup and then we have $G_\alpha G_\beta = x_\alpha G_\beta = G_\alpha x_\beta = G_{\alpha\beta}$, for every $x_\alpha \in G_\alpha, x_\beta \in G_\beta$. We have also $G_{\alpha\beta\alpha} = G_\alpha G_\beta G_\alpha \subseteq G_\alpha$, i.e. $\alpha\beta\alpha = \alpha$. Hence, A is an anticommutative semigroup ([3] p. 109).

Let x be an arbitrary element of S and let $x \in G_\alpha$. Then we have

$$S\,x\,S = \bigcup_{\beta,\gamma\in A} G_\beta\,x\,G_\gamma = \bigcup_{\beta,\gamma\in A} G_{\beta\alpha\gamma} = \bigcup_{\beta,\gamma\in A} G_{\beta\gamma} \supseteq \bigcup_{\beta\in A} G_\beta = S\,,$$

because $\beta\,\alpha\,\gamma = \beta\,\gamma$ ([3] p. 109). Therefore, there are no proper two-sided ideals in the semigroup S.

We shall now prove that every left principal ideal is a left minimal one. Let $x_\alpha \in G_\alpha$ and $y = y_{\beta\alpha} \in G_{\beta\alpha} = G_\beta\,x_\alpha \subseteq S\,x_\alpha$. Then we have

$$S\,x_\alpha = \bigcup_{\delta\in A} G_\delta\,x_\alpha = \bigcup_{\delta\in A} G_{\delta\alpha} = \bigcup_{\delta\in A} G_{\delta\beta\alpha} = \bigcup_{\delta\in A} G_\delta\,y_{\beta\alpha} =$$
$$= (\bigcup_{\delta\in A} G\,\delta)\,y_{\beta\alpha} = S\,y_{\beta\alpha}\,.$$

Hence, $S\,x$ is a minimal left ideal.

In a similar way, it can be shown that every right principal ideal is a right minimal one.

This proves the Theorem.

P r o o f o f C o r o l l a r y 1. Let G be a group and A an anticommutative semigroup. The direct product $G \times A$ becomes a semigroup if the product is defined by $(x, \alpha)\,(y, \beta) = (x\,y, \alpha\,\beta)$, when $x, y \in G$ and $\alpha, \beta \in A$. If $G_\delta = \{(x, \delta)\,;\, x \in G\}$ then $\{G_\delta\,;\,\delta \in A\}$ is a collection of subgroups of $S = G \times A$, such that the equations (1) and (2) hold. Therefore $G \times A$ is a completely simple semigroup. Then we have $(e, \alpha)\,(e, \beta) = (e, \alpha\,\beta)$ (if e is the neutral element of G), i. e. the set B of idempotents of $G \times A$ is a subsemigroup isomorphic with the semigroup A.

Suppose now, that S is a completely simple semigroup and that the set B of all its idempotents is a subsemigroup. If $F = \{G_\delta\,;\,\delta \in A\}$ is the collection of all maximal (different) subgroups of S, then (as we have seen in the proof of the Theorem) if we put $a_\alpha\,a_\beta \in G_\gamma \Leftrightarrow \alpha\,\beta = \gamma$, A becomes an anticommutative semigroup. Therefore, we have $e_\alpha\,e_\beta = e_\gamma \Leftrightarrow \alpha\,\beta = \gamma$, i. e. $e_\alpha\,e_\beta = e_{\alpha\beta}$, whence it follows that the mapping $e_\alpha \to \alpha$ is an isomorphism of B on A. Let ε be a fixed element of A and let us put $(x_\varepsilon, \alpha)\,\xi = e_\alpha\,x_\varepsilon\,e_\alpha$. Then we have

$$(x_\varepsilon, \alpha)\,\xi = e_\alpha\,x_\varepsilon\,e_\alpha = e_\alpha\,x_\varepsilon\,e_\varepsilon\,e_\alpha = e_\alpha\,x_\varepsilon\,e_{\varepsilon\alpha} = x_\varepsilon\,\varphi_{\varepsilon,\,\varepsilon\alpha}\,\psi_{\varepsilon\alpha,\,\alpha}\,,$$

whence, by the proof of Lemma 1.8, we find that ξ is an one-to-one mapping of $G \times A$ onto S. We also have (using Lemma 1.7)

$$[(x_\varepsilon, \alpha)\,(y_\varepsilon, \beta)]\,\xi = (x_\varepsilon\,y_\varepsilon, \alpha\,\beta)\,\xi = e_{\alpha\beta}\,x_\varepsilon\,y_\varepsilon\,e_{\alpha\beta} =$$
$$= e_\alpha\,e_{\varepsilon\beta}\,e_\varepsilon\,x_\varepsilon\,y_\varepsilon\,e_\varepsilon\,e_{\alpha\beta}\,e_\beta =$$
$$= e_\alpha\,e_\varepsilon\,x_\varepsilon\,y_\varepsilon\,e_\varepsilon\,e_\beta = e_\alpha\,x_\varepsilon\,y_\varepsilon\,e_\beta =$$
$$= e_\alpha\,x_\varepsilon\,e_{\varepsilon\alpha\beta\varepsilon}\,y_\varepsilon\,e_\beta = e_\alpha\,x_\varepsilon\,e_\varepsilon\,e_\alpha\,e_\beta\,e_\varepsilon\,y_\varepsilon\,e_\beta =$$
$$= e_\alpha\,x_\varepsilon\,e_\alpha \cdot e_\beta\,y_\varepsilon\,e_\beta = (x_\varepsilon, \alpha)\,\xi \cdot (y_\varepsilon, \beta)\,\xi\,,$$

i. e. ξ is an isomorphism of $G \times A$ on S. This completes the proof of Corollary 1.

Clearly, Corollary 2 is a consequence of the Theorem, of Lemma 1.7 and of Corollary 1.

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O POTPUNO PROSTIM POLUGRUPAMA

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Sadržaj

Neka je G grupa, L i R dva neprazna skupa, a λ preslikavanje skupa $R \times L$ u G. Ako stavimo

$$(x_1 ; l_1, r_1)(x_2 ; l_2, r_2) = (x_1 \lambda(r_1, l_2) x_2 ; l_1, r_2) \quad (x_i \in G, l_i \in L, r_i \in R),$$

onda skup $S = G \times L \times R$ postaje polugrupa[1], koju ćemo označiti sa $S(G; L, R; \lambda)$. Za polugrupu se kaže da je potpuno prosta bez nule ako je izomorfna sa nekom polugrupom oblika $S(G; L, R; \lambda)$; u daljem izlaganju izostavljaćemo izraz »bez nule«.

Poznato je više karakterističnih osobina potpuno prostih polugrupa (videti na primer [3], gl. V), a osnovni zadatak ovog rada je da se ukaže na još jednu karakteristiku pomenute klase polugrupa. Upravo, pokazano je da je polugrupa S potpuno prosta ako i samo ako postoji neka familija podgrupa $F = \{G_\alpha ; \alpha \in A\}$, takva da su zadovoljeni sledeći uslovi: (*i*) $S = \cup G_\alpha$, (*ii*) $G_\alpha G_\beta G_\alpha \subseteq G_\alpha$, za svaki par $G_\alpha, G_\beta \in F$. U tom su slučaju međusobno disjunktne i izomorfne sve grupe koje pripadaju datoj familiji; osim toga imamo i $G_\alpha G_\beta \in F$, ako je $G_\alpha, G_\beta \in F$.

Ako je G grupa i A antikomutativna polugrupa, t. j. tačan je identitet $\alpha\beta\alpha = \alpha$ u polugrupi A, onda je direktni proizvod $G \times A$ potpuno prosta polugrupa. Pri tome je u skupu $G \times A$ operacija određena sa $(x, \alpha)(y, \beta) = (xy, \alpha\beta)$. Potpuno prosta polugrupa S je tog oblika, t. j. izomorfna sa direktnim proizvodom neke grupe G

[1] Polugrupom nazivamo algebarsku strukturu sa jednom binarnom asocijativnom operacijom.

i antikomutativne polugrupe A, ako i samo ako je potpolugrupa polugrupe S skup svih idempotentnih elemenata te polugrupe. U tom slučaju je polugrupa A izomorfna sa polugrupom idempotentnih elemenata.

Na kraju, kao posledica prethodnih rezultata, dobija se i sledeći. Ako je polugrupa S unija neke familije podgrupa od kojih je svaka i levi ideal,[2] onda je S izomorfna nekoj polugrupi oblika $G \times A$, gde je G grupa A neprazan skup, a operacija određena sa $(x, \alpha)(y, \beta) = (xy, \beta)$.

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[2] Q je levi ideal polugrupe S ako je $SQ \subseteq Q$.

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ON SEMIGROUPS S IN WHICH EACH PROPER SUBSET Sx IS A GROUP

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A semigroup S is called *F_{ep}-semigroup* if there is a proper subset Sx of S (i. e. $Sx \neq S$) and if every such subset is a subgroup. Clearly, every F_e-semigroup [3] (and, therefore, every F-semigroup [2]) is an F_{ep}-semigroup too. The purpose of this paper is to describe the structure of F_{ep}-semigroups.

1. Let S be an F_{ep}-semigroup and let us put:

$$P = \{x\,;\, Sx = S\},\ Q = \{x\,;\, x \in Sx \neq S\},\ B = \{x\,;\, x \notin Sx \neq S\}.$$

Then we have: $S = Q \cup B \cup P$ and $Q \cap B = Q \cap P = B \cap P = 0$. The set Q is non-empty; if Q is not a group then it is an F_{ep}-semigroup; at least one of the sets B, P is non-empty, when Q is a group.

1.1. There is a group G and a set A such that Q is isomorphic with the semigroup $G \times A$ defined by:

$$(x, \alpha)\,(y, \beta) = (xy, \beta) \quad (x, y \in G\,;\, \alpha, \beta \in A). \tag{1}$$

Therefore, we may suppose that $Q = G \times A$. If $G_\alpha = \{(x, \alpha)\,;\, x \in G\}$, then $\{G_\alpha\,;\, \alpha \in A\}$ is a collection of all the left ideals of S which are subgroups; and these groups are isomorphic with G.

1.2. Let us suppose that the set B is non-empty. Then, there is a mapping $(\varphi(\mathrm{b}), \xi_b)$ of B in $Q\,(= G \times A)$ such that the following statements are satisfied:

$$(x, \alpha)\,\mathrm{b} = (x\,\varphi(\mathrm{b}), \xi_b)\,; \tag{2}$$

$$\mathrm{b}\,(x, \alpha) = (\varphi(\mathrm{b})\,x, \alpha)\,; \qquad (x \in G\,;\, \alpha \in A\,;\, \mathrm{b}, \mathrm{c} \in B) \tag{3}$$

$$\mathrm{b}\,\mathrm{c} = (\varphi(\mathrm{b})\,\varphi(\mathrm{c}), \xi_c). \tag{4}$$

1.3. If the set P is non-empty then it is a left simple subsemigroup (i. e. $Pp = P$, for each $p \in P$) and moreover, $xp \in P \Leftrightarrow x \in P$. Then there is a homomorphism ψ of P in G and a mapping $\tau : (\alpha, \mathrm{p}) \to \alpha p$ of $A \times P$ in A such that

$$\alpha\,(\mathrm{p}\,\mathrm{q}) = (\alpha\,\mathrm{p})\,\mathrm{q}\,; \tag{5}$$

$$A\,\mathrm{p} = A\,; \tag{6}$$

$$(x, \alpha)\,\mathrm{p} = (x\,\psi(\mathrm{p}), \alpha\,\mathrm{p})\,; \ (x \in G\,;\, \alpha \in A\,;\, \mathrm{b} \in B\,;\, \mathrm{p}, \mathrm{q} \in P) \tag{7}$$

$$p\,(x, \alpha) = (\psi(\mathrm{p})\,x, \alpha)\,; \tag{8}$$

$$p\,\mathrm{b} = (\psi(\mathrm{p})\,\varphi(\mathrm{b}), \xi_b). \tag{9}$$

1.4. Let $B \neq 0, P \neq 0$ and let us put

$$a \in p[b] \Leftrightarrow a p = b. \tag{10}$$

Then we have

For each $p \in P$ and $b \in B$, $p[b]$ is a non-empty subset of B; (11)

$$p[q[b]] = p q[b] \text{ (where } p[q[b]] = \bigcup_{c \in p[b]} p[c]); \tag{12}$$

$$b \neq c \Rightarrow p[b] \cap p[c] = 0; \tag{13}$$

$$a \in p[b] \Rightarrow \varphi(b) = \varphi(a)\psi(p) \;\&\; \xi_b = \xi_a p. \tag{14}$$

If $a p \notin B$, then we have

$$a p = (\varphi(a)\psi(p), \xi_a p). \tag{15}$$

If there is an idempotent in P (or if the set B is finite) then in each subset $p[b]$ there is only one element, and $\{p[b]; p \in P\}$ is a collection of permutations of the set B. Therefore (in that case) the equation $xp = b$ is uniquely solvable in B, for every $p \in P$, $b \in B$.

2. Let G be a group, P a left simple semigroup and $A (\neq 0)$, B two sets such that $G \times A, B$ and P are mutually disjoint. Some of the sets P, B may be empty, but at least one of them is non-empty if A contains only one element. Let $(\varphi(b), \xi_b)$ be a mapping of B into $G \times A$, ψ a homomorphism of P in G and $\tau: (a, p) \to a p$ a mapping of $A \times P$ into A, such that the statements (5) and (6) are satisfied. If the sets B and P are non-empty, then let $\Gamma = \{p[b]; p \in P, b \in B\}$ be a collection of subsets of B such that the statements (11)—(14) are satisfied.

2.1. Let $S = S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma) = G \times A \cup B \cup P$, and let the product of two elements of S be defined by (2)—(4), (7)—(10) and by (15) if $a \notin p[B]$; if $p, q \in P$ then $pq = s$ in $P \Leftrightarrow pq = s$ in S. Then $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ is an F_{ep}-semigroup.

2.2. For an arbitrary group G, left simple semigroup P and sets $A (\neq 0)$, B there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ -semigroup. Namely it is sufficient to put $\varphi(b) = u, \xi_b = \gamma, \psi(p) = e, a p = a$, for every $b \in B, p \in P, a \in A$, where $u \in G$ and $\gamma \in A$ are fixed elements, and e is the identity of the group G.

For an arbitrary group G, a left simple semigroup without idempotents P, and a non-empty set A, there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$-semigroup such that (for each $p \in P$ and $b \in B$) $p[b]$ contains only one element and $p[B] = \bigcup_{b \in B} p[b]$ is a proper subset of B.

I do not know whether there are semigroups $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ such that some of the subsets $p[b]$ contain more than one element.

3. From **1.** and **2.** it follows:

T h e o r e m. *Every F_{ep}-semigroup is isomorphic with some $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$-semigroup.*

4. A semigroup S is called an F_{em}-semigroup if there is a proper minimal left ideal and if all such ideals are subgroups. Clearly, every F_{ep}-semigroup is an F_{em}-semigroup too.

4.1. Let S be an F_{em}-semigroup such that $SS = Q$, where $Q (= G \times A)$ is the class sum of all the minimal left ideals; and let the set $B = S \setminus Q$ be non-empty. Then there are mappings $\varphi (b)$ of B in G and $\xi (\alpha, b)$ of $A \times B$ in A such that the following statements are valid:

$$b (x, \alpha) = (\varphi (b) x, \alpha) ; \tag{16}$$

$$(x, \alpha) b = (x \varphi (b), \xi (\alpha, b)) ; \tag{17}$$

$$a b = (\varphi (a) \varphi (b), \xi (\xi (\alpha, a), b)) ; (x \in G, b \in B, \alpha, \beta \in A) \tag{18}$$

$$\xi (\xi (\alpha, a), b) = \xi (\xi (\beta, a), b) . \tag{19}$$

Therefore $\xi (\xi (\alpha, a), b)$ is a mapping of $B \times B$ into A.

4.2. Let G be a group and $A (\neq 0), B$ two sets. Let φ be a mapping of B into G and $\xi (\alpha, b)$ of $A \times B$ into B, such that (19) holds. If the product of two elements of $S = G \times A \cup B$ is defined by (1), (3), (17) and (18), then S becomes an F_{em}-semigroup.

4.3. Let G be a group and u a fixed element of G, and let $A = \{\alpha, \beta, \gamma\}, B = \{b\}$. If we put $\xi (\alpha, b) = \xi (\beta, b) = \alpha, \xi (\gamma, b) = \beta, \varphi (b) = u$, then (19) is satisfied, and therefore $S = G \times A \cup \{b\}$ is an F_{em}-semigroup. But S is not an F_{ep}-semigroup because $Sb = G \times A \neq S$ is not a subgroup of S.

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O POLUGRUPAMA S U KOJIMA JE SVAKI PRAVI PODSKUP OBLIKA Sx PODGRUPA

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Sadržaj

Nazovimo *F_{ep}-polugrupom* svaku polugrupu S sa osobinom da su svi pravi podskupovi oblika Sx grupe.

Ako je S F_{ep}-polugrupa i ako stavimo $Q = \{x ; x \in Sx \neq S\}$, $B = \{x ; x \notin Sx \neq S\}, P = \{x ; Sx = S\}$, dobijamo $S = Q \cup B \cup P$, pri čemu su skupovi Q, B i P međusobno disjunktni. Daćemo sada opis strukture F_{ep}-polugrupa.

1°. Q je levi ideal i on je (kao polugrupa) izomorfan sa nekom polugrupom oblika $G \times A$, gde je G grupa, A neki skup i operacija određena sa (1). Zato možemo smatrati da je $Q = G \times A$.

2°. Postoji preslikavanje $b \to (\varphi(b), \xi_b)$ skupa B u Q koje zadovoljava jednačine (2)—(4).

3°. P je potpolugrupa polugrupe S. Pri tome imamo $Pp = P$, $Qp = Q$ i $B \subseteq Bp \subseteq B \cup Q$ za svako $p \in P$. Postoji F_{ep}-polugrupa gde je $B \subset Bp$ i takva gde je $B = Bp \subset B \cup Q$.

4°. Postoji homomorfizam ψ polugrupe P u grupi G i preslikavanje $(a, p) \to a p$ skupa $A \times P$ u A, koji zadovoljavaju jednačine (5)—(9). Ako za $a \in B, p \in P$ imamo $a p \in Q$ onda je tačna i jednačina (15).

5°. Prema 4°, za svaki par $p \in P, b \in B$ postoji bar jedno rješenje jednačine $x p = b$ koje se nalazi u skupu B. Ako je $p[b]$ skup svih takvih rešenja onda familija $\Gamma = \{p[b]\,;\, p \in P, b \in B\}$ tih podskupova skupa B zadovoljava uslove (12)—(14).

Ako postoji idempotentan elemenat u polugrupi P, ili ako je konačan skup B, onda $p[b]$ sadrži samo jedan elemenat, t. j. jednačina $x p = b$ je jednoznačno rešiva. Pri tome imamo $Bp = B$ i $p[B] = \bigcup_{b \in B} p[b] = B$. Postoji F_{ep}-polugrupa u kojoj je (za svako $p \in P$) $p[B]$ pravi podskup skupa B, t. j. bar jedan elemenat skupa B nije rešenje jednačine oblika $x p = b$. Nije poznat primer F_{ep}-polugrupe u kojoj neka takva jednačina ima više od jednog rešenja.

Jasno je da neki od skupova Q, B i P može biti prazan. Pri tome imamo: (i) $G \times A = 0 \Rightarrow B = 0, P = S \neq 0$, a u ovom slučaju je $Sx = S$, za svako $x \in S$; (ii) $B = 0$ ili $P = 0 \Rightarrow \Gamma = 0$.

6°. Neka je G grupa, P polugrupa sa osobinom $Pp = P$ za svako $p \in P$ i neka su B i A dva skupa; pri tome pretpostavljamo da su međusobno disjunktni skupovi $G \times A, P, B$ i dopuštamo da je neki od njih prazan, ali treba da je zadovoljen gore pomenuti uslov (i). Neka je $b \to (\varphi(b), \xi_b)$ preslikavanje skupa B u $G \times A, \psi$ homomorfizam polugrupe P u grupi G i $(a, p) \to a p$ preslikavanje skupa $A \times P$ u A i neka su pri tome zadovoljene jednačine (5)—(6). Neka je $\Gamma = \{p[b]\,;\, p \in P, b \in B\}$ familija podskupova skupa B koja zadovoljava uslove (12)—(14).

Skup $S = G \times A \cup B \cup P$ postaje F_{ep}-polugrupa ako se operacija odredi sa: (i) $p_1 p_2 = p$ u $P \Leftrightarrow p_1 p_2 = p$ u S; (ii) jednačinama (1)—(4), (7)—(9) i (10) ili (15) u zavisnosti od toga da li je $a \in p[B]$ ili $a \notin p[B]$.

Prema tome, sa 1°—5° je potpuno opisana struktura klase F_{ep}-polugrupa.

F_{ep}-polugrupe čine potklasu klase polugrupa u kojima su svi minimalni levi ideali i podgrupe. U radu je dat primer polugrupe koja nije F_{ep}-polugrupa iako su svi minimalni levi ideali grupe.

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