Proposition 4.3. Each $\underline{H} \in Massin$ is infinite. □

Proposition 4.4. Every groupoid $\underline{H} \in Massin$ contains infinitely many subgroupoids that are not injective.

Namely, if b is a base in $\underline{H}$, then for any $i \geq 2$, $Q_i = \{b^n \mid n \geq i\}$ is a subgroupoid of $\underline{H}$ and $\underline{Q}_i \notin Massin$. □

Proposition 4.5. *Massfr* is a proper subclass of *Massin*.

Proof. Let A be an infinite set and let $H = A \times \mathbf{N}$. Instead of $(a, n) \in H$ we will write a^n, and moreover, a instead of a^1. The fact that A is infinite implies that A, H and

$$C = \{(a^m, b^n) \mid a, b \in A,\ a \neq b,\ m, n \in \mathbf{N}\},)$$

have the same cardinality. Let $\varphi : C \to H$ be an injective mapping and define a groupoid $\underline{H} = (H, \bullet)$ as follows:

$$(\forall a, b \in A,\ a \neq b,\ m, n \in \mathbf{N}) \quad a^m \bullet a^n = a^{m+n}, \qquad a^m \bullet b^n = \varphi(a^m, b^n).$$

Then $\underline{H} \in Massin$.

Namely, $a = \beta(a^k)$, $k = \varepsilon(a^k)$, for each $a \in A$, $k \in \mathbf{N}$. And, if $a^m, b^n \in H$, $a \neq b$, then $a^m \bullet b = \varphi(a^m, b^n)$ is a base that is not prime in $\underline{H}$. The injectiveness of φ implies that the condition (ii) of the definition holds as well. Then, $H \setminus im(\varphi)$ is the set of primes in $\underline{H}$. Therefore, if φ is bijective, then the set of primes in $\underline{H}$ is empty, and then $\underline{H} \notin Massfr$. □

Proposition 4.6. If $\underline{H} \in Massin$ is such that there exist at least two distinct base elements in $\underline{H}$, then the set of base elements in $\underline{H}$ is infinite.

Proof. Let b, c be base elements in $\underline{H}$ and $b \neq c$. Then, $\{b^k c \mid k \geq 1\}$ is an infinite set of base elements in $\underline{H}$. □

As a corollary we obtain the following.

Proposition 4.7. If $\underline{H} \in Massin$, then the following conditinons are equivalent:

(a) $\underline{H}$ is commutative;

(b) $\underline{H}$ is associative;

(c) $\underline{H}$ is isomorphic to the additive semigroup of positive integers;

(d) There is only one base element in $\underline{H}$;

(e) $\underline{H} \in Massfr$ with one-element basis. □

Below we assume that $\underline{H} \in Massin$, $\underline{Q}$ is a subgroupoid of $\underline{H}$ and the following notation:

$$\beta(H) = \{\beta(a) \mid a \in H\}, \quad C = Q \cap \beta(H),$$

$$D = \{b \in \beta(H) \setminus Q \mid (\exists a \in Q)\, b = \beta(a)\},$$

$$r_b = \min\{k \mid b^k \in Q\}, \quad \text{where } b \in D.$$

Proposition 4.8. If $D = \emptyset$, then $\underline{Q} \in Massin$.

Proof. This is a consequence from the definition of *Massin*. □

Proposition 4.9. If $D \neq \emptyset$, then the following statements are true.

1) For every $b \in D$, the element b^{r_b} is prime in $\underline{Q}$.

2) If, for every $b \in D$, $b^s \in Q$ implies $r_b \mid s$, then $\underline{Q} \in Massin$.

3) If there are $b \in D$ and $s \in \mathbf{N}$ such that r_b does not devide s and $b^s \in Q$, and if s is the least integer with this property, then b^s is prime in $\underline{Q}$ and $\underline{Q} \notin Massin$.

Proof. 1) If b^r $(r = r_b)$ were not prime in $\underline{Q}$, then we would have $b^r = b^i b^j$ for some $b^i, b^j \in Q$, $i + j = r$, and this contradicts the choice of r.

2) Suppose that $a \in Q$ is such that $b = \beta(a) \in D$. By 1), b^r $(r = r_b)$ is the base of a in $\underline{Q}$ and the exponent of a in $\underline{Q}$ is $\varepsilon(a)|r$. Thus $\underline{Q} \in Massin$.

3) Let $s = \min\{k \in \mathbf{N} \mid b^k \in Q$ and r does not devided $k\}$. Then b^s is prime in $\underline{Q}$. (Namely, if b^s were not prime, then we would have $b^s = b^i b^j$ for some $b^i, b^j \in Q$, $(i + j = r)$. By 1), $r \mid i$ and $r \mid j$, which implies $r \mid s$, a contradiction with the choice of s.) Thus the elements b^r, b^s are prime in $\underline{Q}$. Since $(b^r)^s = b^{r+s} = (b^s)^r$, we have that b^{r+s} has two distinct bases in $\underline{Q}$, and thus $\underline{Q} \notin Massin$. □

As a corollary of Propositions 4.8–4.9, we obtain.

Proposition 4.10. $\underline{Q} \notin Massin$ iff there is $b \in \beta(H)$ and $r, s \in \mathbf{N}$ such that $2 \leq r < s$ and b^r, b^s are prime in $\underline{Q}$. □

5. Bruck Theorem for the variety of monoassociative groupoids

Below we show the following proposition, analogous to Proposition 1.1, that we call **Bruck Theorem** for the variety of monoassociative groupoids ([4]).

Proposition 5.1. A groupoid $\underline{H} \in Mass$ is free in $Mass$ iff the following two conditions are satisfied:

(a) $\underline{H} \in Massin$.

(b) The set B of primes in $\underline{H}$ generates $\underline{H}$.

Proof. If $\underline{H} \in Massfr$ then, by Proposition 4.5, $\underline{H} \in Massin$, and, by Proposition 3.3, the set B of primes generates $\underline{H}$.

Let $\underline{H} \in Massin$ and the set B of primes generates $\underline{H}$.

If $B = \{b\}$, then $H = \{b^n \mid n \geq 1\}$, and b is the unique base element in $\underline{H}$ and, by Proposition 4.2, $\underline{H}$ is free in $Mass$ with the basis $\{b\}$.

It remains the case when B contains at least two distinct elements. As in §4 we denote by $\beta(H)$ the set of bases in $\underline{H}$. Clearly, each prime in $\underline{H}$ belongs to $\beta(H)$, and thus $B = B_0 \subseteq \beta(H)$. By (ii) of the definition of injectiveness, we also have $B_1 \subseteq \beta(H)$, where

$$B_1 = \{a^m b^n \mid a, b \in B_0,\ a \neq b,\ m, n \in \mathbf{N}\}.$$

Assume that: $B_0, B_1, \ldots, B_k$ are nonempty sets of bases such that $B_i \cap B_j = \emptyset$ if $i \neq j$. Define B_{k+1} by:

$$B_{k+1} = \{c^m d^n \mid m, n \in \mathbf{N},\ c \neq d,\ \{c,d\} \subseteq B_0 \cup \ldots \cup B_k,\ \{c,d\} \cap B_k \neq \emptyset\}.$$

By (ii) of the definition, we have $B_{k+1} \subseteq \beta(H)$, $B_{k+1} \neq \emptyset$ and $B_{k+1} \cap B_i = \emptyset$, for each $i \in \{1, 2, \ldots, k\}$. Moreover, the fact that $B (= B_0)$ generates $\underline{H}$ implies that

$$\beta(H) = \cup\{B_s \mid s \geq 0\}.$$

If

$$B_i^{\wedge} = \{\alpha^s \mid \alpha \in B_i,\ s \in \mathbf{N}\},$$

then $i \neq j$ implies $B_i^{\wedge} \cap B_j^{\wedge} = \emptyset$ and

$$H = \cup\{B_i^{\wedge} \mid i \geq 1\}.$$

Let $\underline{G} \in Mass$ and $\lambda : B \to G$. Define a sequence of mappings $\varphi_i : B_i^{\wedge} \to G$ as follows:

$$b \in B_0,\ n \geq 1 \Rightarrow \varphi_0(b^s) = (\lambda(b))^s;$$

$$c^m d^n \in B_1,\ n \geq 1 \Rightarrow \varphi_1((c^m d^n)^s) = ((\varphi_0(c))^m (\varphi_0(d))^n)^s;$$

$$c^m d^n \in B_{k+1},\ c \in B_i, d \in B_j \Rightarrow \varphi_{k+1}((c^m d^n)^s) = ((\varphi_i(c))^m (\varphi_j(d))^n)^s.$$

Then, the union $\varphi = \cup_{k=0}^{\infty} \varphi_k$ is a homomorphism of $\underline{H}$ into $\underline{G}$ that extends the given mapping $\lambda : B \to G$. □

Below we assume that $\underline{H} = (H, \cdot) \in Massfr$, $\underline{Q}$ is a subgroupoid of $\underline{H}$ and B is the set of primes (i.e. B is the basis) of $\underline{H}$.

Using the fact that any groupoid $\underline{H} = (H, \cdot) \in Massfr$ with the basis B is isomorphic with the groupoid $\underline{R}$ constructed in §3, and the statements (3.3) and (3.4), we can state the following

Proposition 5.2. There exist a mapping $x \mapsto |x|$ of H into $\mathbf{N}$, and a mapping $x \mapsto \mathrm{cn(x)}$ of H into the set L_B of all finite nonempty subsets of B, such that

$$1) \quad |b| = 1, \quad |xy| = |x| + |y|,$$

$$2) \quad \mathrm{cn}(b) = \{b\}, \quad \mathrm{cn}(xy) = \mathrm{cn}(x) \cup \mathrm{cn}(y),$$

for any $b \in B$, $x, y \in H$. □ [2]

Proposition 5.3. The set P of primes in $\underline{Q}$ is nonempty and generates $\underline{Q}$.

Proof. Assume that $p \in Q$ is such that

$$|p| = \min\{|x| \mid x \in Q\}.$$

Then p is a prime in $\underline{Q}$, and thus the set P of primes in $\underline{Q}$ is nonempty.

Denote by $\underline{T}$ the subgroupoid of $\underline{Q}$ generated by P and assume that for each $a \in Q$ such that $|a| \leq k$, we have $a \in T$. (In the case $|a| = 1$, we have $a \in P$.) Then, if $d \in Q$ is such that $|d| = k + 1$, we have: $d \in T$ if $d \in P$, and if $d \in Q \setminus P$, then there exist $b, c \in Q$ such that $d = bc$. Then, by Proposition 5.2.1), $|b|, |c| \leq k$, and therefore $b, c \in T$, which implies that $d \in T$. □

As a corollary of Propositions 4.8–4.9, Proposition 5.1. and Proposition 5.3, we obtain the following characterization of free subgroupoids of groupoids in $Massfr$.

[2] Note that the existence of such mappings can be shown without using the free groupoid $\underline{R}$. Namely, the fact that $(\mathbf{N}, +) \in Mass$ implies that there exists a homomorphism $|\ | : H \to \mathbf{N}$ such that $|b| = 1$ for each $b \in B$. Also, the fact that $(L_B, \cup) \in Mass$ implies that there is a homomorphism $\mathrm{cn} : H \to L_B$, such that $\mathrm{cn}(b) = \{b\}$ for each $b \in B$.

Proposition 5.4. If $\underline{H} \in Massfr$ and $\underline{Q}$ is a subgroupoid of $\underline{H}$, then the following conditions are equivalent:

(a) $\underline{Q} \in Massin$;

(b) $\underline{Q} \in Massfr$;

(c) There are no prime elemenents b^r, b^s in $\underline{Q}$, where b is a base in $\underline{H}$ and $2 \leq r < s$. □

A corollary of Proposition 4.2 is the following

Proposition 5.5. If $\underline{H} \in Massfr$ is with one-element basis and $\underline{Q}$ is a subgroupoid of $\underline{H}$, then: $\underline{Q} \in Massfr$ iff $\underline{Q}$ is cyclic. □

Proposition 5.6. Let $\underline{H} \in Massfr$ with the two-element basis $B = \{a, b\}$ and $\underline{Q}$ be the subgroupoid of $\underline{H}$ generated by

$$C = \{a^k b^k \mid k \in \mathbf{N}\}.$$

Then $\underline{Q} \in Massfr$ with the infinite basis C.

Proof. The assumption $a \neq b$ implies that each element $c \in C$ is a base in $\underline{H}$; moreover, $a^m b^m = a^n b^n$ implies $m = n$, i.e. the set C is infinite. Note that, by (3.4), $(\forall t \in Q)(\mathrm{cn}(t) = \{a, b\})$, and thus $a^k, b^k \notin Q$. Therefore, every $c \in C$ is prime in $\underline{Q}$ and, by Proposition 5.4 (c), $\underline{Q} \in Massfr$. □

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ЗА МОНОАСОЦИЈАТИВНИТЕ ГРУПОИДИ

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Резиме

Предмет на оваа работа е многуобразието (означено со *Mass*) од моноасоцијативни групоиди, т.е. групоиди во кои секој цикличен подгрупоид е полугрупа. Даден е опис на слободните објекти во *Mass*. Користејќи соодветна дефиниција на поимот инјективен групоид во *Mass*, се покажува дека еден групоид $\underline{H}$ е слободен во *Mass* ако и само ако $\underline{H}$ е инјективен во *Mass* и множеството прости елементи во $\underline{H}$ го генерира $\underline{H}$. (Ова својство е наречено Теорема на Брак за *Mass*.) Ниедна од класите *Massin* (т.е. класата инјективни објекти во *Mass*) и *Massfr* (т.е. класата слободни објекти во *Mass*) не е наследна. Добиена е карактеризација на слободните подгрупоиди од еден групоид $\underline{H} \in Massfr$ и покажано е дека секој групоид $\underline{H} \in Mass$ со двоелементна база има подгрупоид $\underline{Q} \in Massfr$ со бесконечна база.

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