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MULTIVARIATE INTEGRATION IN WEIGHTED SOBOLEV SPACES

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. A formula for the mean square worst-case error of the multivariate integration in the weighted Sobolev space $H_{Sob,s,\gamma}$ is given in the terms of Walsh functions over finite groups. A new version of the weighted diaphony, the so-called $(W(b), \gamma)-$diaphony, is proposed as a quantitative measure for uniform distribution of sequences in $[0, 1)^s$. It is proved that the computing complexity of the $(W(b), \gamma)-$diaphony of an arbitrary net composed of N points in $[0, 1)^s$ is $O(N^2)$. The mean square worst-case error of the multivariate integration in the space $H_{Sob,s,\gamma}$ and the weighted $(W(b), \gamma)-$diaphony are connected.

1. Introduction

Following Aronszajn [1], we will recall the concept of reproducing kernels for Hilbert spaces. Let F be a space of functions defined on E forming a Hilbert space. The function $K(x, y)$ of $x, y \in E$ is called a reproducing kernel for the space F if the following properties hold: for every fixed $y \in E$ the function $K(x, y)$ considered as a function of x belongs to F; **(reproducing property)** for every function $f \in F$ and every $y \in E$ the equality $f(y) = \langle f(x), K(x, y)\rangle_x$ holds, where the subscript x indicates that the inner product is given with respect to the variable x.

Let $s \geq 1$ be a fixed integer and s will denote the dimension everywhere in the paper.

Let $H_s(K)$ be a Hilbert space with reproducing kernel $K : [0, 1)^{2s} \to \mathbf{R}$ and a norm $|| \cdot ||_{H_s(K)}$. We are interested in approximating the multivariate integral $I_s(f) = \int_{[0,1)^s} f(\mathbf{x})d\mathbf{x},\ \ f \in H_s(K)$. Let $N \geq 1$ be an arbitrary fixed integer. We will approximate the integral $I_s(f)$ through quasi-Monte Carlo algorithm with equal quadrature weights $Q_s(f; P_N) = \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)$, where $P_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ is a deterministic sample point net composed of N points in $[0, 1)^s$. The **worst-case error** of the integration in the space $H_s(K)$ by using the quasi-Monte Carlo algorithm $Q_s(f; P_N)$ is defined as

$$e(H_s(K); P_N) = \sup_{f\in H_s(K),||f||_{H_s(K)}\leq 1} |I_s(f) - Q_s(f; P_N)| .$$

The multivariate weighted Hilbert spaces are a weighted tensor product of univariate Hilbert spaces. The weights in these tensor products model the dependence of the functions on different coordinates. Sloan and Woźniakowski [11] propose to arrange the arguments $x_1, x_2, \ldots, x_s$ of the integrands in such a way that x_1 is the most important

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coordinate, x_2 is the next one, and so on. It is realized by associating non-increasing positive real weights $\gamma_1, \gamma_2, \dots, \gamma_s$ to the successive coordinate direction, so we have a vector of weights $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_s > 0$. The motivation for the choice of the vectors of weights is that in practice the weights guarantee the tractability of the quasi-Monte Carlo algorithm, which is the main requirement of this method for multivariate integration. In practice the tractability does not exist in unweighted spaces.

The weighted Hilbert spaces can be divided into two main classes: weighted Korobov spaces and weighted Sobolev spaces. Sloan and Woźniakowski [12] consider three variants of the classical Sobolev spaces related to the integration of non-periodic and periodic functions of these spaces. The multivariate weighted Sobolev spaces are weighted tensor products of univariate Sobolev spaces which consist of absolutely continuous functions whose first derivatives are square integrable.

Following Kuipers and Niederreiter [9] we will recall the concept for uniformly distributed sequences. Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N \geq 1$ and an arbitrary subinterval J of $[0,1)^s$ with a volume $\mu(J)$, we denote by $A_N(\xi; J)$ the number of the points $\mathbf{x}_n$ of the sequence ξ whose indices n satisfy the inequalities $0 \leq n \leq N-1$, and belong to the interval J. The sequence ξ is called uniformly distributed in $[0,1)^s$ if the equality $\lim_{N\to\infty} N^{-1} A_N(\xi; J) = \mu(J)$ holds for every subinterval J of $[0,1)^s$.

In general, the discrepancy and the diaphony are quantitative measures for uniform distribution of sequences in $[0,1)^s$. Zinterhof [14] uses the trigonometric functional system to introduce the "classical" diaphony. Grozdanov and Ristovska [6] propose the weighted version of the classical diaphony, the so-called weighted diaphony and the worst-case error of the integration in weighted Korobov spaces is presented in the terms of the weighted diaphony. The different kinds of the diaphony which is based on using orthonormal functional systems in base $b \geq 2$, as the dyadic diaphony introduced by Hellekalek and Leeb [8], the $b-$adic diaphony introduced by Grozdanov and Stoilova [7] have been considered so far as quantitative measures for uniform distribution of sequences. First Dick and Pillichshammer [3] connect the worst-case error of the integration in Korobov spaces with the $b-$adic diaphony. Grozdanov [5] introduced the weighted version of the $b-$adic diaphony, the so-called weighted $b-$adic diaphony and presented the worst-case error of the integration in weighted Korobov spaces in the terms of the weighted $b-$adic diaphony.

The **purposes** of our paper are:

1. To give a formula for the mean square worst-case error of the integration in the weighted Sobolev spase $H_{Sob,s,\gamma}$ in the terms of Walsh functions over finite groups.

2. To give the concept of the weighted $(W(b),\gamma)-$diaphony as a quantitative measure for uniform distribution of sequences in $[0,1)^s$.

3. To give the relation between the mean square worst-case error of the integration in the weighted Sobolev space $H_{Sob,s,\gamma}$ and the weighted $(W(b),\gamma)-$diaphony.

The paper has the following organization: In section 2 the concept of the Walsh functions over finite groups is reminded. In section 3 the multivariate integration in weighted Sobolev space $H_{Sob,s,\gamma}$ is investigated. In section 4 the concept of the weighted $(W(b),\gamma)-$diaphony is given. In section 5 the proof of Theorem 2 is exposed. In section 6 the proof of Theorem 4 is presented.

2. Walsh functions over finite groups

As a main tool of our work we will use the Walsh functional system over finite abelian groups. Following Larcher, Niederreiter and W. Ch. Schmid [10] we will recall the concept of this functional system. For a given integer $m \geq 1$ let $\{b_1, \ldots, b_m : 1 \leq l \leq m, b_l \geq 2\}$ be a set of fixed integers. For $1 \leq l \leq m$ let $\mathbf{Z}_{b_l} = \{0, 1, \ldots, b_l - 1\}$, $\oplus_{b_l}$ be the operation summation mod b_l of the elements of the set $\mathbf{Z}_{b_l}$ and $(\mathbf{Z}_{b_l}, \oplus_{b_l})$ is the discrete cyclic group of order b_l. Let $G = \mathbf{Z}_{b_1} \times \ldots \times \mathbf{Z}_{b_m}$ and for each pair $\mathbf{g} = (g_1, \ldots, g_m) \in G$ and $\mathbf{y} = (y_1, \ldots, y_m) \in G$ let us set $\mathbf{g} \oplus_G \mathbf{y} = (g_1 \oplus_{b_1} y_1, \ldots, g_m \oplus_{b_m} y_m)$. Then, $(G, \oplus_G)$ is a finite abelian group of order $b = b_1 b_2 \ldots b_m$. For the defined base b let us denote $\mathbf{Z}_b = \{0, 1, \ldots, b-1\}$ and let $\varphi : \mathbf{Z}_b \to G$ be an arbitrary bijection with the condition $\varphi(0) = \mathbf{0}$. For $\mathbf{g}, \mathbf{y} \in G$ let the character $\chi_{\mathbf{g}}(\mathbf{y})$ be defined as $\chi_{\mathbf{g}}(\mathbf{y}) = \prod_{l=1}^{m} \exp\left(2\pi\mathbf{i}\frac{g_l y_l}{b_l}\right)$.

Definition 1. For an arbitrary integer $k \geq 0$ with the $b-$adic representation $k = \sum_{i=0}^{\nu} k_i b^i$, where for $0 \leq i \leq \nu$ $k_i \in \{0, 1, \ldots, b-1\}$ and $k_\nu \neq 0$ and a real $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $i \geq 0$ $x_i \in \{0, 1, \ldots, b-1\}$ and for infinitely many values of i $x_i \neq b-1$, the function ${}_{G,\varphi}wal_k : [0,1) \to \mathbf{C}$ is defined in the following way ${}_{G,\varphi}wal_k(x) = \prod_{i=0}^{\nu} \chi_{\varphi(k_i)}\left(\varphi(x_i)\right)$.

The set $W_{G,\varphi} = \{{}_{G,\varphi}wal_k : k = 0, 1, \ldots\}$ is called the Walsh functional system over the finite group G with respect to the bijection φ.

Let $\mathbf{N}_0$ be the set of non-negative integers. For a vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbf{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we set ${}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{G,\varphi}wal_{k_j}(x_j)$. For any dimension $s \geq 1$ the system $\{{}_{G,\varphi}wal_{\mathbf{k}} : \mathbf{k} \in \mathbf{N}_0^s\}$ is a complete orthonormal functional system on $L_2([0,1)^s)$.

In the case when $m = 1$, $G = \mathbf{Z}_b$ and $\varphi = id-$ the identity of the set $\mathbf{Z}_b$ in itself, the obtained system $W_{\mathbf{Z}_b,id}$ is the system $W(b)$ of the Walsh functions in base b, defined by Chrestenson [2]. The set $W(2)$ is the original Walsh [13] functional system. So, for each integer $k \geq 0$ with the $b-$adic representation $k = \sum_{i=0}^{\nu} k_i b^i$, where for $0 \leq i \leq \nu$ $k_i \in \{0, 1, \ldots, b-1\}$ and $k_\nu \neq 0$ and a real $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, the Walsh function in base b ${}_bwal_k : [0,1) \to \mathbf{C}$ is defined as ${}_bwal_k(x) = \prod_{i=0}^{\nu} \exp\left(2\pi\mathbf{i}\frac{x_i k_i}{b}\right)$.

For arbitrary $\mu, \nu \in \mathbf{Z}_b$ let us set $\mu \oplus_{G,\varphi}^{b} \nu = \varphi^{-1}(\varphi(\mu) \oplus_G \varphi(\nu))$. For arbitrary reals $x, y \in [0,1)$ with $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$ and $y = \sum_{i=0}^{\infty} y_i b^{-i-1}$ let us set $x \oplus_{b,G,\varphi} y = \sum_{i=0}^{\infty} \left[x_i \oplus_{G,\varphi}^{b} y_i\right] b^{-i-1}$ and $x \oplus_b y = \sum_{i=0}^{\infty} \left[x_i + y_i (\bmod\ b)\right] b^{-i-1}$. For arbitrary vectors $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ and $\mathbf{y} = (y_1, \ldots, y_s) \in [0,1)^s$ we set $\mathbf{x} \oplus_{s,b,G,\varphi} \mathbf{y} = (x_1 \oplus_{b,G,\varphi} y_1, \ldots, x_s \oplus_{b,G,\varphi} y_s)$ and $\mathbf{x} \oplus_{s,b} \mathbf{y} = (x_1 \oplus_b y_1, \ldots, x_s \oplus_b y_s)$.

We will define the procedures "digital shift" in base b over the group G with respect to the bijection φ and "digital shift" in base b. For an arbitrary real $x \in [0,1)$ and a fixed real $\sigma \in [0,1)$ the reals $y = x \oplus_{b,G,\varphi} \sigma$ and $y = x \oplus_b \sigma$ we will call respectively a "$(b, G, \varphi)-$digitally shifted point" and a "$b-$adic digitally shifted point". For an arbitrary vector $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ and a fixed vector $\sigma = (\sigma_1, \ldots, \sigma_s) \in [0,1)^s$ we define respectively a "$(s, b, G, \varphi)-$digitally shifted vector" $\mathbf{y} = \mathbf{x} \oplus_{s,b,G,\varphi} \sigma$ and a "$(s, b)-$digitally shifted vector" $\mathbf{y} = \mathbf{x} \oplus_{s,b} \sigma$ as $\mathbf{y} = (x_1 \oplus_{b,G,\varphi} \sigma_1, \ldots, x_s \oplus_{b,G,\varphi} \sigma_s)$ and $\mathbf{y} = (x_1 \oplus_b \sigma_1, \ldots, x_s \oplus_b \sigma_s)$. For a given net $P_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ and a vector $\sigma \in [0,1)^s$ we will call respectively the nets $P_N((s, b, G, \varphi); \sigma) = \{\mathbf{x}_0 \oplus_{s,b,G,\varphi} \sigma, \mathbf{x}_1 \oplus_{s,b,G,\varphi} \sigma, \ldots, \mathbf{x}_{N-1} \oplus_{s,b,G,\varphi} \sigma\}$ and $P_N((s, b); \sigma) = \{\mathbf{x}_0 \oplus_{s,b} \sigma, \mathbf{x}_1 \oplus_{s,b} \sigma, \ldots, \mathbf{x}_{N-1} \oplus_{s,b} \sigma\}$ a "$(s, b, G, \varphi)-$digitally shifted point net" and a "$(s, b)-$digitally shifted point net".

3. Multivariate integration in the weighted Sobolev spaces $H_{Sob,s,\gamma}$

Let $H_s(K)$ be a Hilbert space with a reproducing kernel $K(\mathbf{x}, \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in [0,1)^s$, a inner product $\langle \cdot, \cdot \rangle_{H_s(K)}$ and a norm $|| \cdot ||_{H_s(K)}$. Since $H_s(K)$ is a subset of $L_2([0,1)^s)$, then for an arbitrary fixed $\mathbf{y} \in [0,1)^s$ the function $K(\mathbf{x}, \mathbf{y})$ considered as a function of $\mathbf{x}$

belongs to $L_2([0,1)^s)$. We additionally assume that $K(\mathbf{x},\mathbf{y}) \in L_1([0,1)^{2s})$. We will give the next definitions:

Definition 2. For an arbitrary reproducing kernel K we define respectively the "associated $(s,b,G,\varphi)-$digitally shifted function" as

$$K_{(s,b,G,\varphi)-\mathrm{ds}}(\mathbf{x},\mathbf{y}) = \int_{[0,1)^s} K(\mathbf{x} \oplus_{s,b,G,\varphi} \sigma, \mathbf{y} \oplus_{s,b,G,\varphi} \sigma) d\sigma, \mathbf{x},\mathbf{y} \in [0,1)^s$$

and the "associated $(s,b)-$digitally shifted function"

$$K_{(s,b)-\mathrm{ds}}(\mathbf{x},\mathbf{y}) = \int_{[0,1)^s} K(\mathbf{x} \oplus_{s,b} \sigma, \mathbf{y} \oplus_{s,b} \sigma) d\sigma, \mathbf{x},\mathbf{y} \in [0,1)^s.$$

Following [1] it is easy to prove that the functions $K_{(s,b,G,\varphi)-\mathrm{ds}}$ and $K_{(s,b)-\mathrm{ds}}$ are reproducing kernels, too.

Definition 3. Let $H_s(K)$ be an arbitrary Hilbert space generated by the kernel K. Let P_N be an arbitrary net composed of N points in $[0,1)^s$. We define respectively the notions a mean square worst-case error $e_{s,b,G,\varphi}(H_s(K);P_N)$ and $e_{s,b}(H_s(K);P_N)$ of the integration in the space $H_s(K)$ by using a random $(s,b,G,\varphi)-$ digitally shifted and $(s,b)-$digitally shifted point net P_N as

$$e_{s,b,G,\varphi}(H_s(K);P_N) = \left(\int_{[0,1)^s} e^2(H_s(K);P_N((s,b,G,\varphi);\sigma)) d\sigma \right)^{\frac{1}{2}}$$

and

$$e_{s,b}(H_s(K);P_N) = \left(\int_{[0,1)^s} e^2(H_s(K);P_N((s,b);\sigma)) d\sigma \right)^{\frac{1}{2}}.$$

Theorem 1. *For any reproducing kernel K and a point net P_N the mean square worst-case errors $e_{s,b,G,\varphi}(H_s(K);P_N)$ and $e_{s,b}(H_s(K);P_N)$ of the integration in the space $H_s(K)$ satisfy the equality*

$$e_{s,b,G,\varphi}(H_s(K);P_N) = e(H_s(K_{(s,b,G,\varphi)-\mathrm{ds}});P_N)$$

and

$$e_{s,b}(H_s(K);P_N) = e(H_s(K_{(s,b)-\mathrm{ds}});P_N).$$

Following Sloan and Woźniakowski [12] here we will remind and use the concept of the Sobolev space $H_{Sob,s,\gamma}$. For an arbitrary real $\gamma > 0$ let H_γ be the Sobolev space of absolutely continuous real functions defined over $[0,1)$ whose first derivative is square integrable. The inner product in the space H_γ is defined as

$$\langle f,g \rangle_{H_\gamma} = \int_0^1 f(t)dt \int_0^1 g(t)dt + \gamma^{-1} \int_0^1 f'(t)g'(t)dt, \quad \forall f,g \in H_\gamma.$$

The reproducing kernel $K_\gamma(x,y)$ of the space H_γ is given by

$$K_\gamma(x,y) = 1 + \frac{\gamma}{2}\left[B_2(|x-y|) + 2B_1(x)B_1(y)\right], \quad x,y \in [0,1), \tag{1}$$

where for $t \in [0,1)$ $B_2(t) = t^2 - t + \frac{1}{6}$ and $B_1(t) = t - \frac{1}{2}$ are the Bernoulli polynomials respectively of degree 2 and 1.

For an arbitrary vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ of non-increasing positive weights the multivariate Sobolev space $H_{Sob,s,\gamma}$ is defined as a weighted tensor product of the corresponding one-dimensional Sobolev spaces H_{γ_1}, $H_{\gamma_2}, \ldots, H_{\gamma_s}$, so $H_{Sob,s,\gamma} = H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_s}$.

Thus, $H_{Sob,s,\gamma}$ is a Hilbert space with an inner product

$$\langle f,g\rangle_{H_{\mathrm{Sob},s,\gamma}} = \sum_{u\subseteq\{1,2,\ldots,s\}} \prod_{j\in u}\gamma_j^{-1}\int_{[0,1)^{|u|}}\left(\int_{[0,1)^{s-|u|}}\frac{\partial^{|u|}}{\partial\mathbf{x}_u}f(\mathbf{x})d\mathbf{x}_{-u}\right)\times$$
$$\left(\int_{[0,1)^{s-|u|}}\frac{\partial^{|u|}}{\partial\mathbf{x}_u}g(\mathbf{x})d\mathbf{x}_{-u}\right)d\mathbf{x}_u,$$

where for a vector $\mathbf{x}=(x_1,\ldots,x_s)$, x_u denotes the vector of $|u|$ components such that $(x_u)_i = x_i$ for all $i\in u$ and $\mathbf{x}_{-u}$ denotes the vector $\mathbf{x}_{\{1,2,\ldots,s\}\setminus u}$. For $u=\emptyset$ we set $\prod_{j\in u}\gamma_j^{-1}=1$ and for $u=\emptyset$ and $u=\{1,2,\ldots,s\}$ the integral $\int_{[0,1)^0}$ is replaced by 1.

The reproducing kernel of the space $H_{Sob,s,\gamma}$ is given as

$$K_{s,\gamma}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s}K_{\gamma_j}(x_j,y_j),\quad \mathbf{x}=(x_1,\ldots,x_s),\quad \mathbf{y}=(y_1,\ldots,y_s)\in[0,1)^s.$$

In the next theorem the mean square worst-case error of the multivariate integration in the space $H_{\mathrm{Sob},s,\gamma}$ will be presented in the terms of the Walsh functions over finite groups.

Theorem 2. *The mean square worst-case error $e_{s,b,G,\varphi}(H_{\mathrm{Sob},s,\gamma};P_N)$ of the multivariate integration in the weighted Sobolev space $H_{\mathrm{Sob},s,\gamma}$ by using a random $(s,b,G,\varphi)-$digitally shifted point net $P_N=\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}\}$ is given by the equality*

$$e^2_{s,b,G,\varphi}(H_{\mathrm{Sob},s,\gamma};P_N)$$
$$= -1+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{\mathbf{k}\in\mathbf{N}_0^s} r_b(W_{G,\varphi};\gamma;\mathbf{k})_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_n)\overline{_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_h)},$$

for each vector $\mathbf{k}=(k_1,\ldots,k_s)\in\mathbf{N}_0^s$ the coefficient

$$r_b(W_{G,\varphi};\gamma;\mathbf{k}) = \prod_{j=1}^{s} r_b(W_{G,\varphi};\gamma_j;k_j),$$

for each integer $k\geq 0$ and an arbitrary real $\gamma>0$ the coefficient $r_b(W_{G,\varphi};\gamma;k)$ is defined as

$$r_b(W_{G,\varphi};\gamma;k)$$
$$=\begin{cases} 1, \ \text{if}\ \ k=0 \\ -\frac{\gamma}{2}\left\{\frac{1}{3}+\frac{2}{b}\mathrm{Re}\left[\sum_{u=0}^{b-1}\sum_{v=u}^{b-1}(v-u)_{G,\varphi}wal_{k_{a-1}}\left(\frac{v\ominus^b_{G,\varphi}u}{b}\right)\right]\right\}\frac{1}{b^{2a}}, \\ \forall k,\ \ k_{a-1}b^{a-1}\leq k<(k_{a-1}+1)b^{a-1},\ \ a\geq 1,\ \ a\in\mathbf{Z} \\ \ \text{and}\ \ k_{a-1}\in\{1,2,\ldots,b-1\}. \end{cases} \tag{2}$$

Here for an arbitrary complex z the function $Re(z)$ denotes the real part of z.

When in Theorem 2 we replace the system $W_{G,\varphi}$ of the Walsh functions over the group G with respect to the bijection φ with the system $W(b)$ of the Walsh functions in base b we obtain the next corollary:

Corollary 1. The mean square worst-case error $e_{s,b}(H_{\mathrm{Sob},s,\gamma};P_N)$ of the integration in the weighted Sobolev space $H_{\mathrm{Sob},s,\gamma}$ by using a random $(s,b)-$digitally shifted point net $P_N=\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}\}$ is given by the equality

$$e^2_{s,b}(H_{\mathrm{Sob},s,\gamma};P_N) = -1+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{\mathbf{k}\in\mathbf{N}_0^s} r_b(\gamma;\mathbf{k})_b wal_{\mathbf{k}}(\mathbf{x}_n)\overline{_b wal_{\mathbf{k}}(\mathbf{x}_h)},$$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbf{N}_0^s$, $r_b(\gamma; \mathbf{k}) = \prod_{j=1}^s r_b(\gamma_j; k_j)$, for each integer $k \geq 0$ and an arbitrary real $\gamma > 0$ the coefficient $r_b(\gamma; k)$ is defined as

$$r_b(\gamma; k) = \begin{cases} 1, \ \text{if} \ k = 0 \\ \frac{\gamma}{2}\left(\frac{1}{\sin^2 \frac{\pi k_{a-1}}{b}} - \frac{1}{3}\right)\frac{1}{b^{2a}}, \forall k, \ \ k_{a-1}b^{a-1} \leq k < (k_{a-1}+1)b^{a-1}, \\ a \geq 1, \ \ a \in \mathbf{Z}, \ \ \text{and} \ \ k_{a-1} \in \{1, 2, \ldots, b-1\}. \end{cases}$$

The result of Corollary 1 was obtained by Dick and Pillichshammer [4].

4. The $(W(b), \gamma)-$diaphony

We will propose the concept of the weighted $(W(b), \gamma)-$ diaphony.

Definition 4. Let $b \geq 2$ be an arbitrary fixed integer and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_s > 0$ is a vector of real weights. For each integer $N \geq 1$ the **weighted** $(W(b), \gamma)-$**diaphony** $F_N(W(b); \gamma; \xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0, 1)^s$ is defined as

$$F_N(W(b); \gamma; \xi) = \left(\sum_{\mathbf{k}\in\mathbf{N}_0^s, \mathbf{k}\neq\mathbf{0}} r_b(\gamma; \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b wal_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} \in \mathbf{N}_0^s$ the coefficient $r_b(\gamma; \mathbf{k})$ is defined in Corollary 1.

In the next theorem we will give the fact that the weighted $(W(b), \gamma)-$diaphony is a quantitative measure for uniform distribution of sequences.

Theorem 3. *The sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0, 1)^s$ is uniformly distributed in $[0, 1)^s$ if and only if the equality $\lim_{N\to\infty} F_N(W(b); \gamma; \xi) = 0$ holds for an arbitrary vector γ of non-increasing positive weights.*

For arbitrary reals $x, y \in [0, 1)$ we propose the next conditions:
(C1) $x \oplus_b y$ is not a $b-$adic rational;
(C2) x and y are $b-$adic rationals.

The $b-$adic logarithm of $x \in [0, 1)$ will be denoted by $\log_b x$. If x has the $b-$adic representation $x = \frac{x_{g-1}}{b^g} + \frac{x_g}{b^{g+1}} + \ldots$, where for $i \geq g-1$ $x_i \in \{0, 1, \ldots, b-1\}$ and $x_{g-1} \neq 0$, then the integer part of $\log_b x$ is given as $\lfloor \log_b x \rfloor = -g$.

In the next theorem we will give that the computing complexity of the weighted $(W(b), \gamma)-$diaphony of an arbitrary net composed of N points in $[0, 1)^s$ is $O(N^2)$.

Theorem 4. *For an arbitrary real $\gamma > 0$ and $x \in [0, 1]$we define the function*

$$\varphi(b; \gamma; x) = \frac{6b - \gamma}{6b} + \frac{(b+1)\gamma}{6b} b^{\lfloor \log_b x \rfloor} + \frac{\gamma}{2} \sum_{\alpha=1}^{\infty} b^{-2\alpha} \sum_{\beta=1}^{b-1} \frac{1}{\sin^2 \frac{\pi\beta}{b}} \sum_{l=\beta b^{\alpha-1}}^{(\beta+1)b^{\alpha-1}-1} {}_b wal_l(x).$$

For an arbitrary vector of real weights $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_s > 0$, we define the function $\phi(b; \gamma; \mathbf{x}) = -1 + \prod_{j=1}^s \varphi(b; \gamma_j; x_j)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s$. For each integer $N \geq 1$ let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed of N points in $[0, 1)^s$, such that for $0 \leq n \leq N-1$ the coordinates of all points $\mathbf{x}_n$ satisfy the conditions $(C1)$ or $(C2)$, in particular, the coordinates of all points are $b-$adic rationals. Then, the weighted $(W(b), \gamma)-$diaphony $F(W(b); \gamma; \xi_N)$ of the net ξ_N satisfies the equality

$$F^2(W(b); \gamma; \xi_N) = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{h=0}^{N-1} \phi(b; \gamma; \mathbf{x}_n \ominus_{s,b} \mathbf{x}_h).$$

Theorem 5. *Let P_N be an arbitrary net composed of $N \geq 1$ points in $[0,1)^s$. The mean square worst-case error $e_{s,b}(H_{\mathrm{Sob},s,\gamma}; P_N)$ of the multivariate integration in the Sobolev space $H_{\mathrm{Sob},s,\gamma}$ by using a random $(s,b)-$digitally shifted point net P_N and the weighted $(W(b),\gamma)-$diaphony of the net P_N are connected with the equality*

$$e_{s,b}(H_{\mathrm{Sob},s,\gamma}; P_N) = F(W(b);\gamma;P_N).$$

5. Proof of Theorem 2

In the next lemma we will give the Fourier-Walsh presentation of the reproducing kernel $K_{(s,b,G,\varphi)-ds,\gamma}$.

Lemma 1. For an arbitrary real $\gamma > 0$ let $K_\gamma(x,y)$ be the reproducing kernel defined by the equality (1). Then the associated $(b,G,\varphi)-$digitally shifted kernel $K_{(b,G,\varphi)-ds,\gamma}(x,y)$ has a representation of the form

$$K_{(b,G,\varphi)-ds,\gamma}(x,y) = \sum_{k=0}^{\infty} r_b(W_{G,\varphi};\gamma;k)_{G,\varphi}wal_k(x)\overline{_{G,\varphi}wal_k(y)}, \quad \forall x,y \in [0,1),$$

where for each integer $k \geq 0$ the coefficients $r_b(W_{G,\varphi};\gamma;k)$ are defined by the equality (2).

(ii) Let $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_s)$, where $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_s > 0$ be an arbitrary vector of real weights. Then the corresponding multivariate associated $(s,b,G,\varphi)-$digitally shifted kernel $K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x},\mathbf{y})$ has a representation of the form

$$K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbf{N}_0^s} r_b(W_{G,\varphi};\gamma;\mathbf{k})_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x})\overline{_{G,\varphi}wal_{\mathbf{k}}(\mathbf{y})}, \quad \forall \mathbf{x},\mathbf{y} \in [0,1)^s,$$

where for each vector $\mathbf{k} = (k_1,\ldots,k_s) \in \mathbf{N}_0^s$ the coefficient $r_b(W_{G,\varphi};\gamma;\mathbf{k})$ is defined in Theorem 2.

Following Sloan and Woźniakowski [11] we note that the worst-case error of the integration in the space $H_s(K)$ by using the net $P_N = \{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}$ satisfies the equality

$$e^2(H_s(K);P_N)$$

$$= \int_{[0,1)^{2s}} K(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1)^s} K(\mathbf{x}_n,\mathbf{y})d\mathbf{y} + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1} K(\mathbf{x}_n,\mathbf{x}_h). \quad (3)$$

From Theorem 1 the mean square worst-case error satisfies the equality

$$e_{s,b,G,\varphi}(H_{Sob,s,\gamma};P_N) = e(H_{Sob,s,\gamma}(K_{(s,b,G,\varphi)-ds,\gamma});P_N). \quad (4)$$

From (3) and (4) we obtain that

$$e^2_{s,b,G,\varphi}(H_{Sob,s,\gamma};P_N) = \int_{[0,1)^{2s}} K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}$$

$$-\frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1)^s} K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x}_n,\mathbf{y})d\mathbf{y}$$

$$+\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1} K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x}_n,\mathbf{x}_h). \quad (5)$$

According to Lemma 1 we have that $\forall \mathbf{x},\mathbf{y} \in [0,1)^s$

$$K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}\in\mathbf{N}_0^s} r_b(W_{G,\varphi};\gamma;\mathbf{k})_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x})\overline{_{G,\varphi}wal_{\mathbf{k}}(\mathbf{y})}. \quad (6)$$

Using (6) we consecutively have:

$$\int_{[0,1)^{2s}} K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} = r_b(W_{G,\varphi};\gamma;\mathbf{0}) = 1. \tag{7}$$

For each fixed integer n, $0 \le n \le N-1$

$$\int_{[0,1)^{s}} K_{(s,b,G,\varphi)-ds,\gamma}(\mathbf{x}_n,\mathbf{y})d\mathbf{y} = r_b(W_{G,\varphi};\gamma;\mathbf{0}) = 1. \tag{8}$$

From (5), (6), (7) and (8) we obtain the equality

$$e^2_{s,b,G,\varphi}(H_{Sob,s,\gamma};P_N)$$

$$= -1 + \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{\mathbf{k}\in\mathbf{N}_0^s} r_b(W_{G,\varphi};\gamma;\mathbf{k})_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_n)\overline{_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_h)}.$$

Theorem 2 is proved.

6. Proof of Theorem 4

To prove Theorem 4 we need the next preliminary result:

Lemma 2: (i) Let for an arbitrary real $\gamma > 0$ and $x \in [0,1)$ the function $\varphi(b;\gamma;x)$ be defined in Theorem 4. Then for each integer $k \ge 0$ the $k-$th Fourier-Walsh coefficient of the function $\varphi(b;\gamma;\cdot)$ satisfies the equality $\widehat{\varphi}_{W(b)}(b;\gamma;k) = r_b(\gamma;k)$, where the coefficient $r_b(\gamma;k)$ is defined in Corollary 1.

(ii) Let for an arbitrary vector of real weights $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_s)$ and $\mathbf{x} \in [0,1)^s$ the function $\phi(b;\gamma;\mathbf{x})$ be defined in Theorem 4. Then for each vector $\mathbf{k} \in \mathbf{N}_0^s$ the $\mathbf{k}-$th Fourier-Walsh coefficient of the function $\phi(b;\gamma;\cdot)$ satisfies the equality

$$\widehat{\phi}_{W(b)}(b;\gamma;\mathbf{k}) = \begin{cases} 0, & \text{if } \mathbf{k} = \mathbf{0} \\ r_b(\gamma;\mathbf{k}), & \text{if } \mathbf{k} \ne \mathbf{0}, \end{cases}$$

where the coefficient $r_b(\gamma;\mathbf{k})$ is defined in Corollary 1.

In this way we can prove Theorem 4. For the function $\phi(\cdot)$ defined in Theorem 4 we will use the Fourier-Walsh series

$$\phi(b;\gamma;\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{N}_0^s} \widehat{\phi}_{W(b)}(b;\gamma;\mathbf{k})_b wal_{\mathbf{k}}(\mathbf{x}), \quad \forall \;\; \mathbf{x} \in [0,1)^s.$$

In Lemma 2 the Fourier-Walsh coefficients of the function $\phi(b;\gamma;\cdot)$ are calculated. Hence, we obtain that

$$\phi(b;\gamma;\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{N}_0^s,\mathbf{k}\ne\mathbf{0}} r_b(\gamma;\mathbf{k})_b wal_{\mathbf{k}}(\mathbf{x}), \quad \forall \;\; \mathbf{x} \in [0,1)^s.$$

Using the last equality for an arbitrary net $\xi_N = \{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}\}$ composed of N points in $[0,1)^s$ such that for $0 \le n \le N-1$ the coordinates of all points $\mathbf{x}_n$ satisfy the conditions $(C1)$ or $(C2)$, in particular, the coordinates of all points are $b-$adic rationals we consecutively obtain

$$\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\phi(b;\gamma;\mathbf{x}_n\ominus_{s,b}\mathbf{x}_h) = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{\mathbf{k}\in\mathbf{N}_0^s,\mathbf{k}\ne\mathbf{0}} r_b(\gamma;\mathbf{k})_b wal_{\mathbf{k}}(\mathbf{x}_n\ominus_{s,b}\mathbf{x}_h)$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{h=0}^{N-1}\sum_{\mathbf{k}\in\mathbf{N}_0^s,\mathbf{k}\ne\mathbf{0}} r_b(\gamma;\mathbf{k})_b wal_{\mathbf{k}}(\mathbf{x}_n)\overline{_b wal_{\mathbf{k}}(\mathbf{x}_h)}$$

$$= \sum_{\mathbf{k}\in\mathbf{N}_0^s,\mathbf{k}\neq\mathbf{0}} r_b(\gamma;\mathbf{k}) \left| \frac{1}{N}\sum_{n=0}^{N-1} {}_b wal_{\mathbf{k}}(\mathbf{x}_n) \right|^2 = F^2(W(b);\gamma;\xi_N).$$

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