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ON A METHOD OF SOLVING THE PROBLEM FOR A THIRD ORDER EQUATION WITH MULTIPLE CHARACTERISTICS

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. In the paper, the periodical solution of the second boundary value problem is constructed for the third order equation with multiple characteristics using the reflection method. With the help of obtained solutions, the inhomogeneous problem is reduced to the homogeneous problem.

1. Introduction

The third order equation with multiple characteristics

$$U_{xxx} - U_{yy} = 0 \qquad (1)$$

was considered first in works by H.Block [1] and E. Del Vecchio [2, 3]. Using superpositions, specially selected elementary solutions and asymptotical methods, fundamental solutions of the equation (1) were constructed in the form [4]:

$$U(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} f(t), \quad V(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} \varphi(t),$$

where

$$f(t) = \frac{3}{2} t^{\frac{1}{2}} \int\limits_t^\infty \tau^{-\frac{3}{2}} f^*(\tau) d\tau + c^+, \ t > 0, \ f(t) = \frac{3}{2}|t|^{\frac{1}{2}} \int\limits_{-\infty}^t |\tau|^{-\frac{3}{2}} f^*(\tau) d\tau + c^-, \ t < 0,$$

$$\varphi(t) = \frac{3}{2}|t|^{\frac{1}{2}} \int\limits_{-\infty}^t |\tau|^{-\frac{3}{2}} \varphi^*(\tau) d\tau + c, \ t < 0,$$

$$f^*(t) = \int\limits_0^\infty \exp\left(-\frac{\lambda^{\frac{3}{2}}}{\sqrt{2}}\right) \cos\left(\frac{\lambda^{\frac{3}{2}}}{\sqrt{2}} + \lambda t\right) d\lambda, \quad -\infty < t < \infty,$$

$$\varphi^*(t) = \int\limits_0^\infty \left[\exp\left(\lambda t - \lambda^{\frac{3}{2}}\right) + \exp\left(-\frac{\lambda^{\frac{3}{2}}}{\sqrt{2}}\right) \sin\left(\frac{\lambda^{\frac{3}{2}}}{\sqrt{2}} + \lambda t\right)\right] d\lambda, \quad t < 0,$$

$t = (x-\xi)|y-\eta|^{-\frac{2}{3}}$, $c^{\pm}$, c are constants.

Then, using these fundamental solutions, various boundary value problems were investigated in [5]–[6].

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In [7], using the similarity method, we constructed fundamental solutions of the equation (1). These solutions have the form

$$U(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} f(t), \quad -\infty < t < \infty,$$

$$V(x,y;\xi,\eta) = |y-\eta|^{\frac{1}{3}} \varphi(t), \quad t < 0$$

where

$$f(t) = \frac{2\sqrt[3]{2}}{\sqrt{3}\pi} t\psi\left(\frac{1}{6},\frac{4}{3};\tau\right), \quad \varphi(t) = \frac{36\Gamma\left(\frac{1}{3}\right)}{\sqrt{3}\pi} t\Phi\left(\frac{1}{6},\frac{4}{3};\tau\right),$$

$$t = \frac{x-\xi}{|y-\eta|^{2/3}}, \quad \tau = \frac{4}{27}t^3;$$

$\psi(a,b;x)$, $\Phi(a,b;x)$ are degenerate hypergeometric functions (see [8]).

For $U(x,y;\xi,\eta)$ the following estimates are valid:

$$\left|\frac{\partial^{k+h}U}{\partial x^h \partial y^k}\right| \le C_{kh}|y-\eta|^{\frac{1-(-1)^k}{2}}|x-\xi|^{-\frac{1}{2}\left[2h+3k-1+\frac{3}{2}(1-(-1)^k)\right]}$$

at $\left|\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right| \to -\infty$ where C_{kh} are constants, $kh = 0,1,2,3,\dots$.

Analogous estimates are valid for $V(x,y;\xi,\eta)$ at $\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}} \to -\infty$.

In [9, 10], we investigated some boundary value problems for the equation (1) in the domain $D = \{p < x < q, 0 < y < l\}$. These formulated problems were studied by the Fourier method. It was necessary in them that boundary data were homogeneous on the bounds of the domain $D: y = 0$ and $y = l$.

2. Statement of the problem

Consider the equation (1) in the domain D where $p > 0$, $q > 0$, $l > 0$ are constants.

Problem A. *To find in the domain D the solution of the equation (1)from the class of* $U(x,y) \in C^{3,2}_{x,y}(D) \cap C^{2,1}_{x,y}(\overline{D})$ *satisfying the following boundary conditions*:

$$U_y(x,0) = \varphi_1(x), \quad U_y(x,l) = \varphi_2(x), \tag{2}$$

$$U(p,y) = \psi_1(y), \quad U(q,y) = \psi_2(y), \quad U_x(q,y) = \psi_3(y), \tag{3}$$

where

$$\varphi_i(x) \in C[p,q],\ i=1,2, \quad \psi_j(y) \in C^3[0,l],\ j=1,2, \quad \psi_3(y) \in C^2[0,l],$$

moreover

$$\varphi_1(p) = \psi_1'(0) = \varphi_1(q) = \psi_2'(0) = \psi_3(0) = 0,$$
$$\varphi_2(p) = \psi_1'(l) = \varphi_2(q) = \psi_2'(l) = \psi_3(l) = 0.$$

3. Uniqueness of the solution

Theorem 1. *The homogeneous problem A has only the trivial solution.*

Proof. Let the homogeneous problem A has a non-trivial solution. Consider the identity

$$\frac{\partial}{\partial x}\left(UU_{xx} - \frac{1}{2}U_x^2\right) - \frac{\partial}{\partial y}(UU_y) + U_y^2 = 0.$$

Integrating this identity in the domain D, taking into account homogeneous boundary conditions, we obtain

$$\frac{1}{2}\int_0^l U_x^2(p,y)dy + \iint_D U_y^2(x,y)dxdy = 0.$$

Hence, $U_y(x,0)=0$, i.e. $U(x,y)=f(x)$.

According to the equation (1) and $U_x(0,y)=0$, we have from the homogeneous boundary condition that $f(x)=0$, then $U(x,y)\equiv 0$. □

4. Existence of the solution

Consider the following subsidiary problem: *to construct the function $v(x,y)$ satisfying the equation (1) and the condition (2) in the domain D.*

Let's construct fundamental solutions for this problem on the segment $(0,l)$. Such a function was constructed for one-dimensional parabolic equations by the reflection method in [11]. Following [11], we represent the function U in the form of the series

$$\overline{Z}(x,y;\xi,\eta) = \sum_{m=-\infty}^{\infty}\left[U(x,2ml+y;\xi,\eta)+U(x,2ml-y;\xi,\eta)\right]. \tag{4}$$

Since U is a fundamental solution, all members of this series satisfy the equation (1). If this series converges uniformly, then the function $\overline{Z}$ also satisfies the equation (1). To prove convergence of the series (4), group its members in the following way:

$$\overline{Z}(x,y;\xi,\eta) = U(x,y;\xi,\eta)+U(x,-y;\xi,\eta)+\sum_{m=1}^{\infty}[U(x,2ml+y;\xi,\eta)+$$

$$+U(x,-2ml+y;\xi,\eta)] + \sum_{m=1}^{\infty}[U(x,2ml-y;\xi,\eta)+U(-x,-2ml-y;\xi,\eta)]. \tag{5}$$

Fixing in D arbitrary points $M_0(x_0,y_0)\neq N_0(\xi_0,\eta_0)$, we obtain the numerical series

$$\overline{Z}(x_0,y_0;\xi_0,\eta_0) = U(x_0,y_0;\xi_0,\eta_0)+U(x_0,-y_0;\xi_0,\eta_0)+S_0+S_1$$

where

$$S_0 = \sum_{m=1}^{\infty}\left[U(x_0,2ml+y_0;\xi_0,\eta_0)+U(x_0,-2ml+y_0;\xi_0,\eta_0)\right],$$

$$S_1 = \sum_{m=1}^{\infty}\left[U(x_0,2ml-y_0;\xi_0,\eta_0)+U(x_0,-2ml-y_0;\xi_0,\eta_0)\right].$$

Let's prove convergence of the series S_0. Application of the integral test of convergence gives

$$S_0 = \int_1^{\infty} U(x_0,2ml+y_0;\xi_0,\eta_0)dm + \int_1^{\infty} U(x_0,-2ml+y_0;\xi_0,\eta_0)dm =$$

$$= \int_1^{\infty} |2ml+y_0-\eta_0|^{\frac{1}{3}} f\left(\frac{x_0-\xi_0}{|2ml+y_0-\eta_0|^{\frac{1}{3}}}\right)dm +$$

$$+\int_1^{\infty} |-2ml+y_0-\eta_0|^{\frac{1}{3}} f\left(\frac{x_0-\xi_0}{|-2ml+y_0-\eta_0|^{\frac{1}{3}}}\right)dm =$$

$$= -\frac{3}{4l}(x_0-\xi_0)^2\left[\int\limits_{\frac{x_0-\xi_0}{|y_0+2l-\eta_0|^{\frac{1}{3}}}}^{0} t^{-3}f(t)dt - \int\limits_{\frac{x_0-\xi_0}{|y_0-2l-\eta_0|^{\frac{1}{3}}}}^{0} t^{-3}f(t)dt\right] =$$

$$= \frac{3}{4l}(x_0-\xi_0)^2 \int\limits_{\frac{x_0-\xi_0}{|y_0-2l-\eta_0|^{\frac{1}{3}}}}^{\frac{x_0-\xi_0}{|y_0+2l-\eta_0|^{\frac{1}{3}}}} t^{-3}f(t)dt.$$

Taking into account the estimate $f(t)$, one can easily be convinced that the last integral converges, as it is a proper integral. This proves convergence of the series S_0, and convergence of the series S_1 and the series composed of partial derivatives, is proved analogously. Hence, the series (4) converges uniformly, therefore the function $\overline{Z}(x,y;\xi,\eta)$ satisfies the equation (1). For the function $\overline{Z}(x,y;\xi,\eta)$ the same estimates are valid as for the function U.

Theorem 2. *The function $\overline{Z}(x,y;\xi,\eta)$ is periodical with the period $2l$ with respect to the argument y, i.e.*

$$\overline{Z}(x,y+2l;\xi,\eta) = \overline{Z}(x,y;\xi,\eta).$$

Proof. Consider $\overline{Z}(x,y+2l;\xi,\eta)$:

$$\overline{Z}(x,y+2l;\xi,\eta) = U(x,y+2l;\xi,\eta) + U(x,-y+2l;\xi,\eta)+$$

$$+\sum_{m=1}^{\infty}[U(x,y+2lm+2l;\xi,\eta) + U(x,y-2lm+2l;\xi,\eta) + U(x,-y+2lm+2l;\xi,\eta)+$$

$$+U(x,-y-2lm+2l;\xi,\eta)] = U(x,y;\xi,\eta) + U(x,-y;\xi,\eta)+$$

$$+\sum_{m=0}^{\infty}[U(x,2l(m+1)+y;\xi,\eta) + U(x,2l(m+1)-y;\xi,\eta)]+$$

$$+\sum_{m=2}^{\infty}[U(x,-2l(m-1)+y;\xi,\eta) + U(x,-2l(m-1)-y;\xi,\eta)] =$$

$$= U(x,y;\xi,\eta) + U(x,-y;\xi,\eta) + \sum_{m_1=1}^{\infty}[U(x,2lm_1+y;\xi,\eta) + U(x,2lm_1-y;\xi,\eta)]+$$

$$+\sum_{m_2=1}^{\infty}[U(x,-2lm_2+y;\xi,\eta) + U(x,-2lm_2-y;\xi,\eta)] = U(x,y;\xi,\eta) + U(x,-y;\xi,\eta)+$$

$$+\sum_{m=1}^{\infty}[U(x,2lm+y;\xi,\eta) + U(x,2lm-y;\xi,\eta) + U(x,-2ml+y;\xi,\eta)+$$

$$+U(x,-2ml-y;\xi,\eta)] = Z(x,y;\xi,\eta).$$

Let's study now properties of the function $\overline{Z}(x,y;\xi,\eta)$ and its derivatives on the bounds of the segment $(0,l)$.

For $\eta=0$ we have

$$\overline{Z}(x,y;\xi,0) = 2U(x,y;\xi,0) + 2\sum_{m=1}^{\infty}[U(x,2lm+y;\xi,0) + U(x,2lm-y;\xi,0)]. \quad (6)$$

For $\eta = l$ we obtain

$$\overline{Z}(x,y;\xi,l) = 2U(x,y;\xi,l) + U(x,-y;\xi,l) + \sum_{m=2}^{\infty} U(x,2lm-y;\xi,l)+$$

$$+\sum_{m=1}^{\infty}[U(x,2lm-y;\xi,l) + U(x,-2ml+y;\xi,l) + U(x,-2ml-y;\xi,l)]. \qquad (7)$$

Calculating the derivative with respect to y and taking into account the relation

$$U_y = U^* sgn(y-\eta),$$

we have

$$\overline{Z}_y(x,y;\xi,\eta) = U^*(x,y;\xi,\eta)sgn(y-\eta) - U^*(x,-y;\xi,\eta)sgn(-y-\eta)+$$

$$+\sum_{m=1}^{\infty}[U^*(x,2lm+y;\xi,\eta)sgn(y+2lm-\eta) - U^*(x,-y+2ml;\xi,\eta)sgn(-y+2ml-\eta)+$$

$$+U^*(x,y+2ml;\xi,\eta)sgn(y+2ml-\eta) - U^*(x,-2ml-y;\xi,\eta)sgn(-2ml-y-\eta)]$$

where

$$U^*(x,y;\xi,\eta) = \frac{1}{|y-\eta|^{\frac{2}{3}}} f^*\left(\frac{x-\xi}{|y-\eta|^{\frac{2}{3}}}\right), \quad f^*(t) = \frac{t}{3\gamma}\psi\left(\frac{7}{6},\frac{4}{3},\frac{4}{27}t^3\right), \quad \gamma = \frac{3\sqrt{3\pi}}{2^{\frac{1}{3}}}.$$

For $\eta = 0$ we have

$$\overline{Z}_y(x,y;\xi,0) = 2U^*(x,y;\xi,0) + 2\sum_{m=1}^{\infty}[U^*(x,2lm+y;\xi,0)-$$

$$-U^*(x,2ml-y;\xi,0)] = 2U^*(x,y;\xi,0) + M(x,y;\xi,0) \qquad (8)$$

where

$$M(x,y;\xi,0) = 2\sum_{m=1}^{\infty}\left[U^*(x,2ml+y;\xi,0) - U^*(x,2ml-y;\xi,0)\right].$$

Then

$$\lim_{y\to 0}\overline{Z}_y(x,y;\xi,0) = 2\lim_{y\to 0}U^*(x,y;\xi,0),$$

i.e.

$$\lim_{y\to 0}M(x,y;\xi,0) = 0.$$

For $\eta = l$ we obtain

$$\overline{Z}_y(x,y;\xi,l) = -2U^*(x,y;\xi,l) + U^*(x,-y;\xi,l) + \sum_{m=1}^{\infty}[U^*(x,2lm+y;\xi,l)-$$

$$-U^*(x,-2ml+y;\xi,l) + U^*(x,-2ml-y;\xi,l)] - \sum_{m=2}^{\infty}U^*(x,2lm-y;\xi,l) =$$

$$= -2U^*(x,y;\xi,l) + N(x,y;\xi,l) \qquad (9)$$

where

$$N(x,y;\xi,l) = U^*(x,-y;\xi,l) - \sum_{m=2}^{\infty}U^*(x,2ml-y;\xi,l)+$$

$$+\sum_{m=1}^{\infty}[U^*(x,2ml+y;\xi,l) - U^*(x,-2ml+y;\xi,l) + U^*(x,-2ml-y;\xi,l)],$$

i.e.

$$\lim_{y\to l} N(x,y;\xi,l) = 0.$$

We conclude from here that

$$\lim_{y\to l} \overline{Z}_y(x,y;\xi,l) = \lim_{y\to l}(-2)U^*(x,y;\xi,l).$$

For $y = l$ we have

$$\overline{Z}_y(x,l;\xi,0) = 2U^*(x,l;\xi,0) + 2\sum_{m=1}^{\infty}[U^*(x,2lm+l;\xi,0) - U^*(x,2ml-l;\xi,0)] = 0,$$

and for $y = 0$ we obtain

$$\overline{Z}_y(x,0;\xi,l) = 0.$$

The function

$$v(x,y) = \frac{1}{2}\int_p^q \overline{Z}(x,y;\xi,0)\varphi_1(\xi)d\xi - \frac{1}{2}\int_p^q \overline{Z}(x,y;\xi,l)\varphi_2(\xi)d\xi \tag{10}$$

will be the function to be found since it satisfies the equation (1) and the condition (3) and has the period $2l$.

Let's prove that (10) is the solution of the subsidiary problem.

Calculating the derivative with respect to y from (10) and taking into account (8) and (9), we have

$$v'_y(x,y) = \frac{1}{2}\int_p^q \overline{Z}_y(x,y;\xi,0)\varphi_1(\xi)d\xi - \frac{1}{2}\int_p^q \overline{Z}_y(x,y;\xi,l)\varphi_2(\xi)d\xi =$$

$$= \frac{1}{2}\int_p^q \left[2U^*(x,y;\xi,0) + M(x,y;\xi,0)\right]\varphi_1(\xi)d\xi -$$

$$-\frac{1}{2}\int_p^q \left[-2U^*(x,y;\xi,l) + N(x,y;\xi,l)\right]\varphi_2(\xi)d\xi =$$

$$= J_1(x,y) + J_2(x,y) + J_3(x,y) + J_4(x,y).$$

Let's consider each expression separately

$$J_1(x,y) = \int_p^q U^*(x,y;\xi,0)\varphi_1(\xi)d\xi = \int_p^q \frac{1}{y^{2/3}} f^*\left(\frac{x-\xi}{y^{2/3}}\right)\varphi_1(\xi)d\xi.$$

Replacing integration variables, we obtain

$$J_1(x,y) = -\int_{\frac{x-p}{y^{2/3}}}^{\frac{x-q}{y^{2/3}}} f^*(t)\varphi_1\left(x - ty^{\frac{2}{3}}\right)dt.$$

Then

$$\lim_{y\to 0} J_1(x,y) = -\lim_{y\to 0}\int_{\frac{x-p}{y^{2/3}}}^{\frac{x-q}{y^{2/3}}} f^*(t)\varphi_1\left(x - ty^{\frac{2}{3}}\right)dt =$$

$$= \int_{-\infty}^{\infty} f^*(t)\varphi_1(x)dt = \varphi_1(x) \int_{-\infty}^{\infty} f^*(t)dt = \varphi_1(x)$$

since (see [8])

$$\int_{-\infty}^{\infty} f^*(t)dt = 1.$$

We have also

$$\lim_{y\to 0} J_2(x,y) = 0$$

since $\lim_{y\to 0} M(x,y;\xi,0) = 0$ and

$$\lim_{y\to 0}[J_3(x,y) + J_4(x,y)] = 0$$

since $\overline{Z}_y(x,0;\xi,l) = 0$.

Hence, we have

$$\lim_{y\to 0} v'_y(x,y) = \varphi_1(x).$$

For $y = l$ we obtain

$$\lim_{y\to l}[J_1(x,y) + J_2(x,y)] = 0$$

since $\overline{Z}_y(x,0;\xi,l) = 0$,

$$J_3(x,y) = \int_p^q U^*(x,y;\xi,l)\varphi_2(\xi)d\xi = \int_p^q \frac{1}{|y-l|^{2/3}} f^*\left(\frac{x-\xi}{|y-l|^{2/3}}\right)\varphi_2(\xi)d\xi.$$

If we replace integration variables, we have

$$J_3(x,y) = -\int_{\frac{x-p}{|y-l|^{2/3}}}^{\frac{x-q}{|y-l|^{2/3}}} f^*(t)\varphi_2\left(x - t|y-l|^{2/3}\right)dt,$$

$$\lim_{y\to l} J_3(x,y) = -\lim_{y\to l} \int_{\frac{x-p}{|y-l|^{2/3}}}^{\frac{x-q}{|y-l|^{2/3}}} f^*(t)\varphi_2\left(x - t|y-l|^{2/3}\right)dt =$$

$$= \int_{-\infty}^{\infty} f^*(t)\varphi_2(x)dt = \varphi_2(x) \int_{-\infty}^{\infty} f^*(t)dt = \varphi_2(x),$$

$$\lim_{y\to l} J_4(x,y) = 0$$

since $\lim_{y\to l} N(x,y;\xi,l) = 0$.

Hence, we obtain

$$\lim_{y\to l} v'_y(x,y) = \varphi_2(x).$$

Thus, the function (10) is the solution of the subsidiary problem in fact, i.e. it satisfies the equation (1), and conditions (2) hold, in addition it has the period $2l$ with respect to y.

One can consider the function $\overline{Z}(x, y; \xi, \eta)$ as the Green function of this subsidiary problem. With the help of the function $v(x, y)$, we can reduce given inhomogeneous boundary conditions on the straight lines $y = 0$ and $y = l$ to homogeneous one.

We look up the solution of problem A in the form

$$W(x, y) = U(x, y) - v(x, y)$$

where $U(x, y)$ is the solution of the problem A and $v(x, y)$ has the form (10). Then we obtain the following problem for the function $W(x, y)$:

$$\begin{cases} W_{xxx} - W_{yy} = 0, \\ W_y(x, 0) = W_y(x, l) = 0, \\ W(p, y) = \overline{\psi}_1(y),\ W(q, y) = \overline{\psi}_2(y),\ W_x(q, y) = \overline{\psi}_3(y). \end{cases} \tag{11}$$

This problem can be solved by the method of separation of variables (see [9]–[10]). Defining the function $W(x, y)$, we find

$$U(x, y) = W(x, y) + v(x, y).$$

□

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