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INEQUALITIES OF DUNKL-WILLIAMS AND MERCER IN QUASI 2-NORMED SPACE

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Abstract. C. Park [3] introduced the term of quasi 2-normed space, and further he has also proved few properties of quasi 2-norm. M. Kir and M. Acikgoz [4] gave the procedure for completing the quasi 2-normed space. Families of quasi-norms generated by quasi 2-norm are considered in [2] and are also proven few statements according to that ones. The inequalities of Dunkl-Williams, Mercer, Pečarić-Rajić and the sharp parallelepiped inequalities are fundamental in the theory of a 2-normed spaces. In quasi 2-normed spaces are proven, [1] and[2], the analogous inequalities of sharp inequalities and inequalities of Pečarić-Rajić type. In this paper will be considered inequalities, which are analogies to Dunkl-Williams and Mercer inequalities in quasi 2-normed spaces.

1. INTRODUCTION

S. Gähler (1965) gave the term of 2-norm ([11]). One of the axioms of 2-norm is the parallelepiped inequality, which is basic one in the theory of 2-normed spaces. Precisously this inequality, analogous as in normed spaces, C. Park has replaced by a new condition, and thus he actually obtained the following definition of quasi 2-normed space:

Definition 1 ([3]). Let L be a real vector space and $\dim L \geq 2$. *Quasi* 2-*norm* is real function $\|\cdot,\cdot\|: L\times L \to [0,\infty)$ such that:

a) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ iff the set $\{x,y\}$ is linearly dependent;

b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$;

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$, and

d) it exists a constant $C\geq 1$ so that $\|x+y,z\|\leq C(\|x,z\|+\|y,z\|)$, holds for all $x,y,z\in L$.

An ordered pair $(L,\|\cdot,\cdot\|)$ is called as *quasi* 2-*normed space*. The smallest possible C such that it satisfies the condition *d*) is called as *modulus of concavity* of quasi 2-norm $\|\cdot,\cdot\|$.

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Further, M. Kir and M. Acikgoz [4] have given few examples of trivial quasi 2-normed spaces and have also considered the question about completing a quasi 2-normed space. In [2] is proven the following Lemma which is one of the basic while proving important inequalities in quasi 2-normed spaces.

Lemma 1. If L is a quasi 2-normed space with modulus of concavity $C \geq 1$, then

$$\| \sum_{i=1}^{n} x_i , z \| \leq C^{1+[\log_2 (n-1)]} \sum_{i=1}^{n} \| x_i , z \| . \tag{1}$$

holds for each $n>1$ and for all $z, x_1, x_2, ..., x_n \in L$. ∎

Further, C. Park gave a characterization of quasi 2-normed space, i.e. proved the following theorem.

Theorem 1 ([3]). Let $(L, \| \cdot, \cdot \|)$ be a quasi 2-normed space. It exists $p,\ 0 < p \leq 1$ and an equivalent quasi 2-norm $||| \cdot, \cdot |||$ over L so that

$$||| x+y, z |||^p \leq ||| x, z |||^p + ||| y, z |||^p , \tag{2}$$

holds for all $x, y, z \in L$. ∎

Definition 2 ([3]). The quasi 2-norm defined in Theorem 1 is called a $(2, p)-$*norm*, and the quasi 2-normed space L is called a $(2, p)-$*normed space*.

2. OUR RESULTS

Theorem 2. Let L be a quasi 2-normed space with modulus of concavity $C \geq 1$, and $V(z)$ be the subspace generated by vector z . The following inequality

$$\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|} , z \| \leq 4C \tfrac{\|x-y,z\|}{\|x,z\|+\|y,z\|} + 2(C-1) \tfrac{\max\{\|x,z\|,\|y,z\|\}}{\|x,z\|+\|y,z\|} , \tag{3}$$

holds true for each $z \in L \setminus \{0\}$ and for all $x, y \in L \setminus V(z)$.

Proof. Let $z \in L \setminus \{0\}$ and $x, y \in L \setminus V(z)$. Since definition 1 we get that

$$\begin{aligned} \| x, z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|} , z \| &= \| x, z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|x,z\|} + \tfrac{y}{\|x,z\|} - \tfrac{y}{\|y,z\|} , z \| \\ &\leq C \| x, z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|x,z\|} , z \| + C \| x, z \| \cdot \| \tfrac{y}{\|x,z\|} - \tfrac{y}{\|y,z\|} , z \| \quad (4) \\ &\leq C \| x-y, z \| + C \,|\, \| y, z \| - \| x, z \| \,| . \end{aligned}$$

Further, once again definition 1implies that

$$\| y, z \| \leq C \| y-x, z \| + C \| x, z \| \text{ and } \| x, z \| \leq C \| x-y, z \| + C \| y, z \| .$$

Therefore,

$$\begin{aligned} \| y, z \| - \| x, z \| &\leq C \| y-x, z \| + (C-1) \| x, z \| \\ &\leq C \| x-y, z \| + (C-1) \max\{ \| x, z \|, \| y, z \| \} \end{aligned}$$

and

$$\| x,z \| - \| y,z \| \le C \| x-y,z \| + (C-1) \| y,z \|$$
$$\le C \| x-y,z \| + (C-1) \max\{\| x,z \|, \| y,z \|\},$$

i.e. the inequality

$$| \| y,z \| - \| x,z \| | \le C \| x-y,z \| + (C-1) \max\{\| x,z \|, \| y,z \|\} . \quad (5)$$

holds true. The inequalities (4) and (5) imply the inequality

$$\| x,z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| \le 2C \| x-y,z \| + (C-1) \max\{\| x,z \|, \| y,z \|\} \quad (6)$$

The following inequality can be proven analogously

$$\| y,z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| \le 2C \| x-y,z \| + (C-1) \max\{\| x,z \|, \| y,z \|\} . \quad (7)$$

Finally, if we add the inequalities (6) and (7) and so obtained inequality is divided by $\| x \| + \| y \| > 0$ we get the inequality (3). ■

Theorem 3. Let L be a $(2,p)-$normed space, $0 < p \le 1$, and $V(z)$ be a subspace generated by vector z. Then

$$\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^p \le 2 \frac{\|x-y,z\|^p + | \|y,z\| - \|x,z\| |^p}{\|x,z\|^p + \|y,z\|^p} , \quad (8)$$

for each $z \in L \setminus \{0\}$ and for all $x, y \in L \setminus V(z)$.

Proof. Definition 2, i.e. the properties of $(2,p)-$norm imply that each $z \in L \setminus \{0\}$ and all $x, y \in L \setminus V(z)$ satisfy the following

$$\| x,z \|^p \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^p = \| x,z \|^p \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|x,z\|} + \tfrac{y}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^p$$
$$\le \| x,z \|^p \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|x,z\|}, z \|^p + \| x,z \|^p \| \tfrac{y}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^p \quad (9)$$
$$\le \| x-y,z \|^p + | \| y,z \| - \| x,z \| |^p$$

and

$$\| y,z \|^p \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^p = \| y,z \|^p \| \tfrac{x}{\|x,z\|} - \tfrac{x}{\|y,z\|} + \tfrac{x}{\|y,z\|} - \tfrac{y}{\|y,z\|}, z \|^p$$
$$\le \| y,z \|^p \| \tfrac{x}{\|x,z\|} - \tfrac{x}{\|y,z\|}, z \|^p + \| y,z \|^p \| \tfrac{x}{\|y,z\|} - \tfrac{y}{\|y,z\|}, z \|^p \quad (10)$$
$$\le \| x-y,z \|^p + | \| y,z \| - \| x,z \| |^p .$$

Finally, if we add the inequalities (9) and (10) and the so obtained inequality we divide by $\| x \|^p + \| y \|^p > 0$, we get the inequality (8). ■

Remark 1. The inequalities (3) and (8) are actually inequalities of Dunkl- Williams type in quasi-normed and $p-$normed space, $0 < p \le 1$, respectively.

Theorem 4. Let L be a quasi 2-normed space with modulus of concavity $C \ge 1$. The following statements are equivalent:

1) For each $z \in L \setminus \{0\}$ and for all $x, y \in L \setminus V(z)$

$$\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| \leq 2C \tfrac{\|x-y,z\|}{\|x,z\|+\|y,z\|} + (C-1) \tfrac{\max\{\|x,z\|,\|y,z\|\}}{\|x,z\|+\|y,z\|}. \qquad (11)$$

2) If $x, y, z \in L$ are such that $\| x,z \| = \| y,z \| = 1$,holds then

$$\| \tfrac{x+y}{2}, z \| \leq C \| (1-t)x + ty, z \| + \tfrac{C-1}{2} \max\{1-t, t\}, \qquad (12)$$

for each $t \in [0,1]$.

Proof. $1) \Rightarrow 2)$. Let assume that 1) is satisfied. Let $x, y, z \in L$ be such that

$$\| x,z \| = \| y,z \| = 1$$

is satisfied. Clearly, for $t = 0$ and $t = 1$, the inequality (12) is satisfied. If $t \in (0,1)$, then 1) implies

$$\begin{aligned}
\| \tfrac{x+y}{2}, z \| &= \tfrac{1-t}{2} (1 + \tfrac{t}{1-t}) \| x+y, z \| \\
&= \tfrac{1-t}{2} (\| x,z \| + \| \tfrac{t}{1-t} y, z \|) \| \tfrac{x}{\|x,z\|} - \tfrac{\frac{t}{t-1} y}{\|\frac{t}{t-1} y, z\|}, z \| \\
&\leq \tfrac{1-t}{2} (\| x,z \| + \| \tfrac{t}{1-t} y, z \|)(2C \tfrac{\|x-\frac{t}{t-1} y, z\|}{\|x,z\|+\|\frac{t}{t-1} y,z\|} + (C-1) \tfrac{\max\{\|x,z\|,\|\frac{t}{t-1} y,z\|\}}{\|x,z\|+\|\frac{t}{t-1} y,z\|}) \\
&= C(1-t) \| x - \tfrac{t}{t-1} y, z \| + \tfrac{(C-1)(1-t)}{2} \max\{1, \tfrac{t}{1-t}\} \\
&= C \| (1-t)x + ty, z \| + \tfrac{C-1}{2} \max\{1-t, t\},
\end{aligned}$$

i.e. the inequality (12) holds true.

$2) \Rightarrow 1)$. Let assume that 2) is satisfied. Further, let x and y be arbitrary non-null vectors at L. Then for $\frac{x}{\|x,z\|}, \frac{-y}{\|y,z\|} \in L$ it is true that

$$\| \tfrac{x}{\|x,z\|}, z \| = \| \tfrac{-y}{\|-y,z\|}, z \| = 1$$

and if $t = \frac{\|y,z\|}{\|x,z\|+\|y,z\|}$, then 2) implies that

$$\begin{aligned}
\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| &= 2 \| \tfrac{\frac{x}{\|x,z\|} + \frac{-y}{\|y,z\|}}{2}, z \| \\
&\leq 2(C \| (1 - \tfrac{\|y,z\|}{\|x,z\|+\|y,z\|}) \tfrac{x}{\|x,z\|} + \tfrac{\|y,z\|}{\|x,z\|+\|y,z\|} \cdot \tfrac{-y}{\|y,z\|}, z \| \\
&\quad + \tfrac{C-1}{2} \max\{1 - \tfrac{\|y,z\|}{\|x,z\|+\|y,z\|}, \tfrac{\|y,z\|}{\|x,z\|+\|y,z\|}\} \\
&= 2C \tfrac{\|x-y,z\|}{\|x,z\|+\|y,z\|} + (C-1) \tfrac{\max\{\|x,z\|,\|y,z\|\}}{\|x,z\|+\|y,z\|}
\end{aligned}$$

i.e. the inequality (11) holds true. ■

Remark 2. The inequality (11) is actually generalization of the inequality

$$\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| \leq \tfrac{2\|x-y,z\|}{\|x,z\|+\|y,z\|},$$

which on 2-normed space, is satisfied if and only if the 2-norm is generated by 2-inner product ([10]). So, it is logically to be stated the following question:

Does the inequality (11) in quasi 2-normed space with modulus of concavity $C \geq 1$ *hold true if and only if it exists a function* $f : L \times L \to \mathbf{R}$ *so that* $f(x, x, z) = \| x, z \|^2$.

References

[1] A. Malčeski and R. Malčeski, *Inequalities of Pečarić-Rajić type in quasi 2-normed space*, IJSIMR, Vol. 3, Issue 5 (2015), 1-7.

[2] A. Malčeski, R. Malčeski, K. Anevska and S. Malčeski, *A remark about quasi 2-normed space*, Applied Mathematical Sciences, Vol. 9, No. 55 (2015), 2717-2727.

[3] C. Park, *Generalized quasi-Banach spaces and quasi-(2;p) normed spaces*, Journal of the Chungcheong Matematical Society, Vol. 19, No. 2 (2006), 197 – 206

[4] M. Kir and M. Acikgoz, *A study involving the completion of quasi 2-normed space*, International Journal of Analysis (2013), http://dx.doi/10.1155.2013/ 512372

[5] R. Malčeski and A. Malčeski, *Inequalities of Dunkl-Williams and Mercer type in quasi-normed space*, IJSIMR, Vol. 3, Issue 1 (2015), 67 – 70

[6] R. Malčeski and K. Anevska, *Families of norms generated by 2-norm*, AJER, Vol. 03, Iss. 05 (2014), 315 – 320

[7] R. Malčeski and K. Anevska, *Parallelepiped inequality into 2-normed space and its consequences*, IJSIMR, Vol. 2, Issue 08 (2014), 719 – 728

[8] R. Malčeski, *Inequalities of Pečarić-Rajić type in quasi-normed space*, Applied Mathematical Sciences, Vol. 9, no. 15 (2015), 729-737

[9] R. Malčeski, *On the Sharp Triangle Inequalities in Quasi-Normed Spaces*, *British Journal of Mathematics & Computer Science*, Vol. 7, No. 4 (2015), 258-265

[10] S. Malčeski, A. Malčeski, K. Anevska, R. Malčeski, *Another characterization's of 2-pre-Hilbert Space*, IJSIMR, Vol. 3, Issue 2 (2015), 45-54

[11] S. Gähler, *Lineare 2-normierte Räume,* Math. Nachr. 28 (1965), 1–42

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