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TOPOLOGICAL GAMES AND TOPOLOGIES ON GROUPS

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Dedicated to Academician Ǵorǵi Čupona

Abstract. In this paper, using the language and techniques of topological games, of sieves and of plumages, we give a conditions for a semitopological group or a paratopological group to be a topological group. We prove that a paratopological group with the Baire property and with given pointwise property is a topological group if and only if it is p-embedded in some pseudocompact space with respective property. The case of n-ary groups is examined too. Some new open problems are formulated.

1. Introduction

By a space we understand a regular topological T_1-space. We use the terminology from [7, 21]. Let $\omega = \{0, 1, 2, ...\}$ and $\mathbb{N} = \{1, 2, ...\}$. By $cl_X H$ we denote the closure of a set H in a space X. A *paratopological* group is a group endowed with a topology such that the multiplication is jointly continuous. Recall that a *semitopological* group is a group with a topology such that the multiplication is separately continuous. A semitopological group with a continuous inverse operation $x \to x^{-1}$ is called a *quasitopological* group.

In 1936 D. Montgomery [26] has proved the following two theorems:

Theorem 1M. *Every completely metrizable separable semitopological group is a topological group.*

Theorem 2M. *Every completely metrizable semitopological group is a paratopological group.*

These two results of D.Montgomery have raised the following general problems:

P1. What additional conditions are needed to be sure that a paratopological group is actually a topological group?

P2. Under what additional conditions does a semitopological group become a paratopological or a topological group?

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In 1957 R.Ellis [20] had established that every locally compact semitopological group is a topological group. Further results on semitopological and paratopological groups were established by Z. Zhelazko [39], N. Brandt [13], L.G. Brown [14], P. Kenderov, I.S. Kortezov, and W.B. Moors [24], E.A. Reznichenko [32], A. Bouziad [10, 11, 12] and other authors (see [7, 2, 3, 22, 23, 29, 30, 33, 37]). Some advances in this direction were made also by the authors in [4].

The problem of A. D. Wallace [36] on the continuity of the inverse in countably compact topological semigroups was studied by many authors too(see [7, 9, 23]).

Various new classes of spaces over which the theorems of D. Montgomery and R. Ellis can be extended were defined either by some natural topological properties, or by requiring that there exists a winning strategy in certain topological games [7, 2, 3, 10, 11, 24, 34]. In the present paper we continue the research from [2, 3, 4, 24] and use topological games to establish some results which are related to the studies in [7, 11, 12, 28]. Some of the results that follow were announced in [5].

Since there exist examples of Hausdorff countably compact paratopological groups which are not topological groups ([30, 33]), the requirement of the regularity of spaces in the main results of the present article is essential.

2. Topological games of Banach-Mazur type

We will consider below several properties $\mathcal{P}$ of sequences of open subsets of a given topological space X. All of these properties will satisfy the following requirements:

C1. If $\{W_n : n \in \mathbb{N}\}$ is a sequence of subsets of the space X with the property $\mathcal{P}$, then any W_n is an open non-empty set and $W_{n+1} \subseteq W_n$ for any $n \in \mathbb{N}$.

C2. If $\{W_n : n \in \mathbb{N}\}$ is a sequence of open subsets and $W_{n+1} \subseteq W_n$ for any $n \in \mathbb{N}$ and M is an infinite subset of $\mathbb{N}$, then the sequence $\{W_n : n \in \mathbb{N}\}$ has the property $\mathcal{P}$ if and only if the subsequence $\{W_n : n \in M\}$ has the property $\mathcal{P}$.

Each property $\mathcal{P}$ determines on the space X a topological game $G_{\mathcal{P}}(X)$ (briefly, $G_{\mathcal{P}}$) which is similar to the Banach-Mazur game. Two players, α and β play a game by selecting non-empty open subsets of X. Player β starts the game by chosing a nonempty open subset U_1 of X. Whenever β chooses an open non-empty subset U_n the player α responds by selecting a non-empty open subset V_n such that $V_n \subseteq U_n$. In turn, player β selects a non-empty open subset $U_{n+1} \subseteq V_n$ and the game goes on. Continuing this procedure indefinitely the players α and β generate a sequence $\{(U_n, V_n) : n \in \mathbb{N}\}$ of open non-empty subsets with the properties $U_{n+1} \subseteq V_n \subseteq U_n$ for $n \in \mathbb{N}$. The sets U_n are the moves of player β and the sets V_n are the moves of α. Every such sequence will be called a *play.* The player α *wins* the play $\{(U_n, V_n) : n \in \mathbb{N}\}$ in the $G_{\mathcal{P}}$-game if the sequence $\{V_n : n \in \mathbb{N}\}$ has the property $\mathcal{P}$. Otherwise the player β is declared to be the winner of this play.

By a *strategy* t for the player α we mean *"a rule"* that specifies each move of the player α in every possible situation. More precisely, the strategy t is a sequence of mappings $\{t_n : n \in \mathbb{N}\}$. The values of the mapping t_n are the moves of player α at the n-th stage of the game. The domains $Dom\, t_n$ of the mappings t_n,

$n \in \mathbb{N}$, are defined inductively. $Dom\, t_1$ consists of all non-empty open subsets U_1 of X. Suppose that $Dom\, t_i$ have already been defined for $i < n$, where $n > 1$. Then $Dom\, t_n$ consists of those n-taples $(U_1, ..., U_n)$ of non-empty open subsets of X such that, for every $i < n$, $(U_1, ..., U_i) \in Dom\, t_i$ and $U_{i+1} \subseteq V_i := t_i(U_1, ..., U_i) \subseteq U_i$.

If the play $\{(U_n, V_n) : n \in \mathbb{N}\}$ has been played according to t (i.e. $V_n = t_n(U_1, ..., U_n)$ for every $n \in \mathbb{N}$), then it is called a *t-play*. A strategy t for the player α is called *a winning strategy* if, player α wins each t-play.

The topological space X is called *$(\alpha, G_{\mathcal{P}})$-favorable* if the player α has a winning strategy in the $G_{\mathcal{P}}(X)$-game.

Similarly, under *a strategy* t for the player β we mean a *"a rule"* that specifies each move of the player β in every possible situation. A strategy t for player β is actually a sequence of mappings $t = \{t_n : n \in \mathbb{N}\}$, where $U_1 = t_1(X)$ is a fixed open non-empty subset of X and the domains and the values of t_n for any $n \geq 2$ satisfy the requirements:

- $U_n = t_n(V_1, ..., V_{n-1})$ is an open non-empty subset of the set V_{n-1} and
- $V_n \subseteq U_n$.

In case the player β applies the strategy t, the generated play $\{(U_n, V_n) : n \in \mathbb{N}\}$ is called *a t-play*. A strategy t for the player β is called *a winning strategy* if β wins every t-play.

A topological space X is called *$(\beta, G_{\mathcal{P}})$-unfavorable* if player β does not have a winning strategy.

A strategy t of one of the players is *stationary* if for any n the function t_n depends only on the last move of the other player. A strategy t of one of the players is *Markov* if for any n the function t_n depends both on the number n and on the last move of the other player.

Definition 2.1. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of a space X has the property BM if $\cap\{H_n : n \in \mathbb{N}\} \neq \varnothing$.

The G_{BM}-game is known under the name *Banach-Mazur game*. It was studied by G. Choquet [18] and some ot the modifications of this game are called *Choquet games*.

It is known (see [28, 15, 18, 31, 34]) that a space X is a Baire space if, and only if, the space X is (β, G_{BM})-unfavorable.

For a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of a space X we put $Lim\{H_n : n \in \mathbb{N}\} = \cap\{cl_X(\cup\{H_m : m > n\}) : n \in \mathbb{N}\}$, i.e. $Lim\{H_n : n \in \mathbb{N}\}$ is the set of all accumulation points of the sequence $\{H_n : n \in \mathbb{N}\}$. If $H_{n+1} \subseteq H_n$ for any $n \in \mathbb{N}$, then $Lim\{H_n : n \in \mathbb{N}\} = \cap\{cl_X H_n : n \in \mathbb{N}\}$.

Definition 2.2. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of the space X is *stable* , or has the property Π, if it satisfies the following conditions:

(S1) $\varnothing \neq U_{n+1} \subseteq U_n$ for any $n \in \mathbb{N}$ and $\cap\{cl_X U_n : n \in \mathbb{N}\} = \cap\{U_n : n \in \mathbb{N}\}$.

(S2) Every sequence $\{V_n : n \in \mathbb{N}\}$ of open sets in X such that $V_n \subseteq U_n$ for each $n \in \mathbb{N}$ and the set $\{n \in \mathbb{N} : V_n \neq \varnothing\}$ is infinite, has an accumulation point in X, i.e. $Lim\{V_n : n \in \mathbb{N}\} \neq \varnothing$.

A subset L of a space X is called *bounded* if for every locally finite family γ of open subsets in X the set $\{U \in \gamma : U \cap L \neq \varnothing\}$ is finite.

A space X is called *feebly compact* if every locally finite family of open subsets in X is finite, i.e. X is bounded in X.

A subset L of a Tychonoff space X is bounded if and only if every continuous function on X is bounded on L (see [7], [21]). For Tychonoff spaces the feeble compactness is equivalent to the pseudocompactness. Every countably compact space is feebly compact.

If a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of the space X is stable (i.e. satisfies the conditions (S1) and (S2)), then $H = \cap\{cl_X U_n : n \in \mathbb{N}\} = Lim\{U_n : n \in \mathbb{N}\}$ is a bounded non-empty subset of the space X.

The $G_{\mathcal{P}}(X)$-game for $\mathcal{P} = \Pi$ will be denoted by $G_\Pi(X)$.

Clearly, every (β, G_Π)-unfavorable space is a Baire space.

Definition 2.3. (see [24, 4, 2]). Let Y be a dense subspace of a space X. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of open subsets of the space X has the property S_Y if $\varnothing \neq H_{n+1} \subseteq H_n$ for any $n \in \mathbb{N}$, any sequence $\{y_n \in Y \cap H_n : n \in \mathbb{N}\}$ has an accumulation point in X and $\cap\{cl_X H_n : n \in \mathbb{N}\} = \cap\{H_n : n \in \mathbb{N}\}$.

Let $S = S_X$. The $G_{\mathcal{P}}(X)$-game for $\mathcal{P} = S$ will be denoted by $G_S(X)$.

Every (β, G_S)-unfavorable space is a Baire space.

A space X is called a strongly Baire space if, and only if, the space X is (β, G_{S_Y})-unfavorable for some dense subspace Y of X (see [24, 15]).

Any sequence with the property S_Y has the property Π.

A sequence $\{H_n : n \in \mathbb{N}\}$ of open subsets of a space X has the property S if and only if it satisfies the following two conditions:

(S3). The set $H = \cap\{H_n : n \in \mathbb{N}\}$ is a non-empty, closed and countably compact subset of X.

(S4) For any open subset $U \supseteq H$ of X there exists $n \in \mathbb{N}$ such that $H \subseteq H_n \subseteq U$ (i.e. $\{H_n : n \in \mathbb{N}\}$ is a base of neighborhoods of H in X).

Definition 2.4. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of a space X has the property k if $H = \cap\{H_n; n \in \mathbb{N}\}$ is a non-empty compact subset and $\{H_n : n \in \mathbb{N}\}$ has property (S4).

Each sequence with the property k has the property S. The $G_{\mathcal{P}}(X)$-game for $\mathcal{P} = k$ will be denoted by $G_k(X)$. Obviously, every (β, G_k)-unfavorable space is a Baire space.

Definition 2.5. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of the space X has the property M if the sequence $\{H_n : n \in \mathbb{N}\}$ is stable and $\cap\{H_n : n \in \mathbb{N}\}$ is a singleton.

If the sequence $\{H_n : n \in \mathbb{N}\}$ has the property M, then it is a base of open neighborhoods for the point $\cap\{H_n : n \in \mathbb{N}\}$.

The $G_{\mathcal{P}}(X)$-game for $\mathcal{P} = M$ will be denoted by $G_M(X)$.

Remark 2.6. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two properties of sequences of open subsets and any sequence with the property $\mathcal{P}_1$ has the property $\mathcal{P}_2$, then we put $\mathcal{P}_1 \leq \mathcal{P}_2$. Thus, by definitions, $M \leq k \leq S \leq S_Y \leq \Pi \leq BM$.

3. α-Favorable and β-unfavorable spaces

Oxtoby theorem (see [28, 34]) can be given another formulation which is more convenient for our considerations.

Theorem 3.1. *Let $\mathcal{P} \leq BM$ be a property of sequences of open subsets of a space X. Suppose the space X does not have the Baire property. Then on X there exists a Markov winning strategy for the player β in the game $G_{\mathcal{P}}(X)$.*

Proof. There exist an open non-empty subset U and a sequence $\{X_n : n \in \mathbb{N}\}$ of nowhere dense closed subsets of X such that $U \subseteq \cup\{X_n : n \in \mathbb{N}\}$. Let $t_1(X) = U \setminus X_1$. For any non-empty open subset V of X and each $2 \leq n \in \mathbb{N}$ fix an open non-empty subset $t_n(V) \subseteq V \setminus X_n$. By construction, $t = \{t_n : n \in \mathbb{N}\}$ is a Markov strategy for the player β which produces plays with empty intersections. Therefore t is winning for player β in $G_{BM}(X)$ and in $G_{\mathcal{P}}(X)$. $\square$

Definition 3.2. Let $\mathcal{P}$ be a topological property of sequences of open sets. A sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open families of X is called a dense-$\mathcal{P}$-sieve on the space X if it has the following properties:

(1ds) The set $X_n = \cup\{U_\alpha : \alpha \in A_n\}$ is dense in X for all $n \in \mathbb{N}$;

(2ds) If $\{H_n : n \in \mathbb{N}\}$ is a sequence of open non-empty subsets, $\cap\{H_n : n \in \mathbb{N}\} \neq \varnothing$, $\{\alpha_n \in A_n\}$ is a sequence of elements and $cl_X H_{n+1} \subseteq H_n \subseteq U_{\alpha_n}$ for any $n \in \mathbb{N}$, then $\{H_n : n \in \mathbb{N}\}$ is a sequence with the property $\mathcal{P}$.

Definition 3.3. Let $\mathcal{P} \leq BM$ be a topological property of sequences of open sets. A sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open families of X is called a complete dense-$\mathcal{P}$-sieve on the space X if it has the following properties:

(1cds) The set $X_n = \cup\{U_\alpha : \alpha \in A_n\}$ is dense in X for all $n \in \mathbb{N}$;

(2cds) If $\{H_n : n \in \mathbb{N}\}$ is a sequence of open non-empty subsets, $\{\alpha_n \in A_n\}$ is a sequence of elements and $cl_X H_{n+1} \subseteq H_n \subseteq U_{\alpha_n}$ for any $n \in \mathbb{N}$, then $\{H_n : n \in \mathbb{N}\}$ is a sequence with the property $\mathcal{P}$;

(3cds) The set $\cup\{V \in \gamma_n : cl_X V \subseteq U_\alpha\}$ is dense in U_α for all $\alpha \in A_n$ and $n \in \mathbb{N}$.

Remark 3.4. If $\{F_n : n \in \mathbb{N}\}$ is a sequence of closed nowhere dense subsets of a space X and $X = \cup\{F_n : n \in \mathbb{N}\}$, then $\{\gamma_n = \{U_n = X \setminus F_n : n \in A_n = \{n\}\} : n \in \mathbb{N}\}$ is a dense-$\mathcal{P}$-sieve on the space X for any topological property $\mathcal{P} \leq BM$ of sequences of open sets. If $\mathcal{P} \leq BM$ and a space X has complete dense-$\mathcal{P}$-sieve, then X is a Baire space.

Remark 3.5. If a space X contains a dense subspace which is a dense G_δ-subset in some feebly compact space, then the space X has a complete dense-Π-sieve. A space with a complete dense-k-sieve contains a dense Čech complete paracompact subspace.

Definition 3.6. A property $\mathcal{P}$ of sequences of non-empty open sets is called *stable*, if whenever $\{H_n : n \in \omega\}$ has property $\mathcal{P}$, then every sequence $\{W_n : n \in \omega\}$ of open non-empty sets such that $W_{n+1} \subseteq W_n \subseteq H_n$, for each $n \in \omega$, also has the property $\mathcal{P}$.

Clearly, the properties Π, S and k are stable. BM is an example of a property which is not stable.

The following theorem and its proof is similar to Theorem 4.3 from [4].

Theorem 3.7. *Let $\mathcal{P}$ be a stable property of sequences of open subsets of a space X and the space X has a dense-$\mathcal{P}$-sieve. The space X is $(\beta, G_{\mathcal{P}})$-unfavorable if and only if X is a Baire space.*

Remark 3.8. Let $\mathcal{P}$ be a property of sequences of open subsets of a space X and $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a dense-$\mathcal{P}$-sieve on X. Then there exist a sequence $\eta = \{\eta_n = \{W_\beta : \beta \in B_n\} : n \in \mathbb{N}\}$ of disjoint families of open non-empty subsets of X and a sequence $\{b_n : B_{n+1} \longrightarrow B_n : n \in \mathbb{N}\}$ of mappings such that:

- η is a dense-$\mathcal{P}$-sieve on X;

- if γ is a complete dense-$\mathcal{P}$-sieve on X, then η is a complete dense-$\mathcal{P}$-sieve on X too;

- $cl_X W_\beta \subseteq W_{b_n(\beta)}$ for all $\beta \in B_{n+1}$ and $n \in \mathbb{N}$;

- the family η_n is a refinement of the family γ_n for any $n \in \mathbb{N}$.

The following assertion was proved in [3] (see also [4] Theorem 5.1 and Theorem 6.4).

Theorem 3.9. *Let $\mathcal{P}$ be a stable property of sequences of open subsets of a space X. The following assertions are equivalent:*

1. The space X is $(\alpha, G_{\mathcal{P}})$-favorable.

2. For the player α there exists a Markov winning strategy in the $G_{\mathcal{P}}(X)$-game.

3. X is a space with a complete dense-$\mathcal{P}$-sieve.

4. Special embeddings of spaces

Let X be a subspace of the space Z. If γ is a family of subsets of Z and $L \subseteq Z$, then $St(L, \gamma) = \cup\{H \in \gamma : H \cap L \neq \varnothing\}$ is the star of the set L with respect to γ. We put $St(x, \gamma) = St(\{x\}, \gamma)$ for any point $x \in Z$.

A sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of families of subsets of the space Z is called:

- a *partial plumage* of X in Z if the sets $U \in \cup\{\gamma_n : n \in \mathbb{N}\}$ are open in Z and for every pair of points $x \in X$ and $z \in Z \setminus X$ there exists $n \in \mathbb{N}$ such that $St(x, \gamma_n) \cap \{x, z\} = \{x\}$;

- a *plumage* of X in Z if γ is a partial plumage and $X \subseteq \cup\gamma_n$ for any $n \in \mathbb{N}$;

- a *star-separation sequence* of X in Z if the sets $U \in \cup\{\gamma_n : n \in \mathbb{N}\}$ are open in Z and for every pair of points $x \in X$ and $z \in Z \setminus X$ there exists $n \in \mathbb{N}$ such that either $St(x, \gamma_n) \cap \{x, z\} = \{x\}$, or $St(z, \gamma_n) \cap \{x, z\} = \{z\}$.

Any partial plumage is a star-separation sequence.

A subspace X of a space Z is called:

- p-embedded in Z if X has a plumage in Z;

- $p^\star$-embedded in Z if X has a partial plumage in Z;

- $p^{\star\star}$-embedded in Z if X has a star-separation sequence in Z.

A space with a plumage in a compact space is called a p-space [1]. A space with a partial plumage in a compact space is called a $p^\star$-space. A space with a star-separation sequence in a compact space is called a $p^{\star\star}$-space (see [4]). Any metrizable space is a p-space. A space X is a paracompact p-space if and only if there exists a perfect mapping of X onto some metrizable space [1].

Remark 4.1. Let X be a $p^\star$-embedded subspace of the space Z. Then for any point $x \in X$ there exists a G_δ-subset H of Z such that $x \in H \subseteq X$. If Z is a $p^\star$-space, then X is a $p^\star$-space too. In particular, any $p^\star$-space is a space of pointwise countable type. A space is of pointwise countable type if each point is contained in some compact subset of countable character.

Remark 4.2. If X is a $p^{\star\star}$-subspace of a $p^{\star\star}$-space Z, then $Z \setminus X$ is a $p^{\star\star}$-space.

Example 4.3. Let X be the Michael line ([21], Example 5.1.32). Then X is a $p^\star$-space and not a p-space. The space X has a σ-disjoint open base and is hereditarily paracompact.

Example 4.4. Let βY be the Stone-Čech compactification of an infinite discrete space Y, $c \in \beta Y \setminus Y$ and $X = Y \cup \{c\}$. The families $\gamma_1 = \{\{x\} : x \in Y\}$ and $\gamma_2 = \{\beta X \setminus \{c\}\}$ form a star-separation sequence of X in βX. Since X is not a space of pointwise countable type, X is not a $p^\star$-sace and it is a $p^{\star\star}$-space.

Remark 4.5. The class $\mathcal{A}$ of $p^{\star\star}$-embedded subspaces of the space Z has the following properties:

- if $X, Y \in \mathcal{A}$, then $X \cap Y \in \mathcal{A}$, $X \cup Y \in \mathcal{A}$, $X \setminus Y \in \mathcal{A}$:

- the class $\mathcal{A}$ contains the open subspaces, the closed subspaces and is closed under A-operation.

Theorem 4.6. *Let $f : X \longrightarrow Y$ be a perfect mapping of a $p^{\star\star}$-space X onto a space Y. Then there exist two subspaces $Y^\star$ and Y° of Y with the properties:*

1. $Y^\star = Y \setminus Y^\circ$.

2. $Y^\star$ is a paracompact p-space with the Baire property.

3. $Y^\star$ is a G_δ-subspace of Y and Y° is a sum of a sequence of closed nowhere dense subsets of Y.

Proof. Obviously, we can assume that the mapping f is irreducible. Let $g : \beta X \longrightarrow \beta Y$ be the continuous extension of the mapping f onto the Stone-Čech compactifications of the spaces X and Y. For any open subset V of βX we put $g^\sharp(V) = Y \setminus g(\beta X \setminus V) = \{y \in \beta Y : g^{-1}(y) \subseteq V\}$. It is well-known that the mapping g is irreducible and the set $g^{-1}(g^\sharp(V))$ is dense in V.

Let $Y' = \cup\{V : V \textit{ is open and of the first category in } Y\}$. Then the set Y' is open and of the first category in Y. Let $H = \beta Y \setminus cl_{\beta Y} Y'$. If the set H is non-empty, then the set $Y \setminus H$ is closed and of the first category in Y and $H \cap Y$ is an open subspace of Y with the Baire property.

If Y' is dense in Y, then $Y = Y'$, $H = \varnothing$ and we put $Y^\circ = Y$, $Y^\star = \varnothing$.

Assume that Y' is not dense in Y. Fix a star-separation sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of X in βX.

If the set $\cup\gamma_n$ is not dense in βX we put $\gamma'_n = \gamma_n \cup \{\beta X \setminus cl_{\beta X} \cup \gamma_n)$. Obviously, $\gamma' = \{\gamma'_n : n \in \mathbb{N}\}$ is a star-separation sequence of X in βX. Therefore, we can assume that the sets $\cup\gamma_n$ are dense in βX.

There exists a sequence $\eta = \{\eta_n = \{V_\beta : \beta \in B_n\} : n \in \mathbb{N}\}$ of disjoint families of open non-empty subsets of βY with the properties:

- for any $n \in \mathbb{N}$ and each $\beta \in B_{n+1}$ there exists a unique $b(\beta) \in B_n$ such that $cl_{\beta Y} V_\beta \subseteq V_{b(\beta)}$;

- for any $n \in \mathbb{N}$ and each $\beta \in B_n$ there exists a $c(\beta) \in A_n$ such that $cl_{\beta Y} V_\beta \subseteq g^\sharp(U_{c(\beta)})$;

- for any $n \in \mathbb{N}$ the set $W_n = \cup\{V_\beta : \beta \in B_n\}$ is dense in βY;

- if $C = \{\beta \in B_1 : g^{-1}(V_\beta \subseteq H\}$, then the set $H_1 = \cup\{g^{-1}(V_\beta : \beta \in C\}$ is dense in the set H.

The set $W_1 = g(H_1)$ is open in βY, $Y \cap W_1$ is a subspace with the Baire property and $Y \setminus W_1$ is a sum of a sequence of closed nowhere dense subsets of Y.

On B_n we consider the discrete topology and put $B = \Pi\{B_n : n \in \mathbb{N}\}$. Let $W = \cap\{W_n : n \in \mathbb{N}\}$. For any $y \in W$ there exists a unique sequence $\varphi(y) = \{\beta_n(y) \in B_n : n \in \mathbb{N}\}$ such that $y \in \cap\{V_{\beta_n(y)} : n \in \mathbb{N}\}$. Obviously, $\beta_n(y) = b(\beta_{n+1}(y))$ for all $n \in \mathbb{N}$. Therefore $\varphi : W \to Z = \varphi(W) \subseteq B$ is a perfect mapping of the space W onto the complete metrizable space Z. By construction, $\varphi^{-1}(\varphi(W \cap V_\beta)) = W \cap V_\beta$. Since the sets W_n are dense and open, $Y^\star = W \cap Y$ is a Baire subspace of Y and $Y^\circ = Y \setminus Y^\star$ is a sum of a sequence of closed nowhere dense subsets of Y.

Let $y \in Y \cap W$ and $w \in W \setminus Y$. Fix $x \in f^{-1}(y)$ and $z \in g^{-1}(w)$. There exists $n \in \mathbb{N}$ such that either $St(x, \gamma_n) \cap \{x, z\} = \{x\}$, or $St(z, \gamma_n) \cap \{x, z\} = \{z\}$. In this case $V_{\beta n(y)} \cap \{y, w\} = \{y\}$. Thus $\varphi(y) \neq \varphi(w)$. Therefore, $\varphi^{-1}(\varphi(Y)) = Y$, $\varphi^{-1}(\varphi(Y^\star)) = Y$ and $\varphi^{-1}(\varphi(Y^\circ)) = Y^\circ$. In particular, $\psi = \varphi|Y^\star$ is a perfect mapping onto a metric space $\varphi(Y^\star)$. $\square$

Corollary 4.7. *([4], Theorem 6.7) Let X be a $p^{\star\star}$-space with the Baire property. Then:*

- X contains a dense G_δ-subspace which is a paracompact p-space with the Baire property;

- if $k \leq \mathcal{P} \leq BM$, then the space X is $(\beta, G_{\mathcal{P}})$-unfavorable.

A mapping $g : X \longrightarrow Y$ is called quasi-perfect if g is continuous, closed and the fibers $f^{-1}(y)$, $y \in Y$, are countably compact.

A space X is called a (complete) M-space if there exists a quasi-perfect mapping onto some (complete) metric space Y (see [27, 17, 38]).

The proofs of the next two theorems are similar to the proof of the Theorem 4.6.

Theorem 4.8. *Let X be a dense subspace of the space Z. If the space X has the Baire property and it is $p^{\star\star}$-embedded in Z, then:*

1. There exists a dense G_δ-subspace Y of Z such that $Y \subseteq X$ and Y is p-embedded in the spaces X and Z.

2. There exists a plumage $\gamma = \{\gamma_n : n \in \mathbb{N}\}$ of Y in Z such that the families γ_n are disjoint.

3. If $k \leq \mathcal{P} \leq BM$ and the space Z is $(\beta, G_{\mathcal{P}})$-unfavorable, then the space X is $(\beta, G_{\mathcal{P}})$-unfavorable too.

4. If the space Z is (β, G_M)-unfavorable, then the space X is (β, G_M)-unfavorable and X contains a metrizable dense G_δ-subspace Y.

Theorem 4.9. *Let X be a dense subspace of the countably compact space Z. If the space X has the Baire property and it is $p^{\star\star}$-embedded in Z, then the space X is $(\beta, G_{\mathcal{P}})$-unfavorable and there exists a dense G_δ-subspace Y of Z such that:*
1. $Y \subseteq X$ and Y is an M-space.
2. There exists a plumage $\gamma = \{\gamma_n : n \in \mathbb{N}\}$ of Y in Z such that the families γ_n are disjoint.

Theorem 4.10. *Let X be a dense subspace of the feebly compact space Z. If the space X has the Baire property and it is $p^{\star\star}$-embedded in Z, then the space X is (β, G_Π)-unfavorable and there exists a dense G_δ-subspace Y of Z for which there exists a plumage $\gamma = \{\gamma_n : n \in \mathbb{N}\}$ of Y in Z such that the families γ_n are disjoint.*

5. Construction of pseudocompact extensions

In this section all spaces are considered to be completely regular and Hausdorff.

Let $\mathcal{P}$ be a stable property of sequences of open sets of spaces.

A space X has the property $\mathcal{P}$ if the sequence $\{U_n = X : n \in \mathbb{N}\}$ has the property $\mathcal{P}$ in X.

A space X is pointwise of type $\mathcal{P}$, or briefly $\mathcal{P}$-pointwise, if for each point $x \in X$ there exists a sequence $\{U_n : n \in \mathbb{N}\}$ with the property $\mathcal{P}$ such that $x \in \cap\{U_n : n \in \mathbb{N}\}$. The pointwise pseudocompact topological groups have been studied in [7].

Following E.Michael [25], a point $x \in X$ is called a *q-point* if there exists a sequence of neighborhoods $\{U_n : n \in \mathbb{N}\}$ of the point x in X such that if $x_n \in U_n$, then the sequence $\{x_n : n \in \mathbb{N}\}$ has a cluster point in X. If any point of the space X is a q-point, then the space X is called a q-space or, in our terminology, an *S-pointwise space.*

If X is a dense subspace of the space Z, then we say that Z is an extension of the space X.

The extension Z of a space X is called:

- a $\mathcal{P}$-extension, if Z is a space with the property $\mathcal{P}$;
- a thin extension if any bounded and closed subset of X is closed in Z as well.

Remark 5.1. For any thin extension Z of a space X the space $\sigma_Z X = X \cup (\beta Z \setminus \cup\{cl_{\beta Z} F : F \textit{ is a bounded subset of } X\})$ is a pseudocompact thin extension of X. Really, if $\sigma_Z X$ is not pseudocompact, then there exist a point $z \in \beta Z \setminus \sigma_Z X$ and a continuous function f on βZ such that $f(z) = 0$ and $f(y) > 0$ for any $y \in \sigma_Z X$. By construction, $z \in cl_{\beta Z} L$ for some bounded subset L of X. Then the function $g(x) = 1/f(x)$ is continuous on X and unbounded on L, a contradiction.

We recall that the G_δ-closure $\omega cl_X Y$ of a set $Y \subseteq X$ in a space is the set of all points $x \in X$ such that every G_δ-set H containing x intersect Y.

Let A be a topological group. Denote by ρA the Raikov completion of a topological group A (see [7]).

Lemma 5.2. *Let A be a topological group, L be a closed bounded subset of A and $F = cl_{\rho A}L$ be a G_δ-subset of the space ρA. Then:*
1. The set L is G_δ-dense in F, i.e. $F = \omega cl_{\rho A}L$.
2. L is a pseudocompact subspace of the space A and $\beta L = F$.

Proof. Assume that $z \in F \setminus \omega cl_{\rho A}L$. Then there exists a G_δ-subset H of ρA such that $z \in H$ and $L \cap H = \varnothing$. Since F is a G_δ-subset, we can suppose that $H \subseteq F$. Then $A \cap H = \varnothing$ and there exists a continuous function f on ρA such that $f(z) = 0$ and $f(y) > 0$ for each $y \in \rho A \setminus H$. The function $g(x) = 1/f(x)$ is continuous on A and unbounded on L, a contradiction. The assertion 1 is proved.

A compact G_δ-subset of a topological group is a Dugundji space ([7], Corollary 10.3.9). Any Dugundji space is perfectly k-normal, i.e. the closure of any open set is a G_δ-subset. Thus the assertion 2 follows from the next lemma. □

Lemma 5.3. *Let Y be a G_δ-dense subspace of a compact perfectly k-normal space X. Then Y is a pseudocompact space and $\beta Y = X$.*

Proof. Follows from ([7], Theorem 6.1.7). □

Let A be a topological group. We put $\sigma A = \sigma_{\rho A}A = A \cup (\beta\rho A \setminus \cup\{cl_{\beta\rho A}F : F \textit{ is a bounded subset of } A\})$. There exists a natural continuous mapping $\varepsilon_A : \beta A \longrightarrow \beta\rho A$ such that $\varepsilon_A(x) = x$ for any $x \in A$.

Assume that $\{U_n : n \in \mathbb{N}\}$ is a sequence of open subsets of A with some stable property $\mathcal{P} \leq BM$. We can suppose that $e \in \cap\{U_n : n \in \mathbb{N}\}$, where e is the identity in A. Obviously, the space A is $\mathcal{P}$-pointwise.

Since A is a dense subgroup of the group ρA, there exists a sequence $v = \{V_n : n \in \mathbb{N}\}$ of open subsets of the space ρA such that:
- $e \in V_{n+1}^2 \subseteq V_n = V_n^{-1}$ for any $n \in \mathbb{N}$;
- $cl_{\rho A}V_{n+1} \subseteq V_n$ for any $n \in \mathbb{N}$;
- for any $n \in \mathbb{N}$ the sets V_{n+1} and $\rho A \setminus V_n$ are functionally separated in ρA;
- $W_n = A \cap V_n \subseteq U_n$ for any $n \in \mathbb{N}$.

By construction, $H = \cap\{V_n : n \in \mathbb{N}\}$ is a closed subgroup of ρA, $P = H \cap A = \cap\{W_n : n \in \mathbb{N}\}$ is a closed subgroup of A and $w = \{W_n : n \in \mathbb{N}\}$ is a sequence of open subsets of A with the property $\mathcal{P}$.

Property 5.1. $H = cl_{\rho A}P = \omega cl_{\rho A}P$.

Proof. Let $O \subseteq \rho A \setminus P$ be an open subset of ρA and $x_0 \in O \cap H$. There exists a sequence $\{O_n : n \in \mathbb{N}\}$ of open subsets of ρA such that $x_0 \in cl_{\rho A}O_{n+1} \subseteq O_n \subseteq O \cap V_n$ for any $n \in \mathbb{N}$. Then $\{L_n = A \cap O_n : n \in \mathbb{N}\}$ is a sequence of open non-empty subsets of A and $cl_A L_{n+1} \subseteq L_n \subseteq W_n$. Since property $\mathcal{P}$ is stable $L = \cap\{L_n : n \in \mathbb{N}\} \neq \varnothing$, $L \subseteq P$ and $L \subseteq H \cap O$, a contradiction. Lemma 5.2 completes the proof. □

Property 5.2. *H is a compact subgroup and for any open in ρA set $V \supseteq H$ there exists $n \in \mathbb{N}$ such that $H \subseteq V_n \subseteq V$.*

Proof. The set P is bounded in A and the closure of a bounded set in a complete uniform space ρA is compact. Thus H is a compact subgroup. Let $V \supseteq H$ be an open subset of ρA and the set $L_n = V_n \setminus clV$ be non-empty for any $n \in \mathbb{N}$. w has the stable property $\mathcal{P}$ and $A \cap L_n \subseteq W_n$, the sequence $\{L_n : n \in \mathbb{N}\}$ has an accumulation point in P, a contradiction. The proof is complete. □

Property 5.3. *ρA is a paracompact Čech complete group.*

Proof. Follows from [16] (see [7], Theorem 4.3.15). □

Let $B^\star = \rho A/H$, $\psi : \rho A \longrightarrow B^\star$ be the natural quotient mapping, $\varphi = \psi|A$ and $B = \psi(A) = \varphi(A)$.

Property 5.4. *The space $B^\star$ is completely metrizable, ψ is an open perfect mapping and φ is an open continuous mapping onto a metrizable space B.*

Proof. Since ρA is is a paracompact Čech complete group and H is a compact subgroup of countable character, $B^\star$ is a completely metrizable space and ψ is an open perfect mapping. From Property 5.1 it follows that φ is an open continuous mapping onto a metrizable space B. □

Property 5.5. *The space σA is G_δ-dense and C-embedded in ρA. In particular, $\sigma A \subseteq \rho A \subseteq \beta\sigma A = \beta\rho A$.*

Proof. From ([7], Theorem 6.9.10) it follows that the spaces A and ρA are Moskow spaces, i.e. the closure of any open set is a union of G_δ-subsets. By construction, $\sigma A = A \cup \psi^{-1}(B^\star \setminus B)$. Thus σA is dense in ρA. Any G_δ-dense subspace of a Moskow space is C-embedded ([7], Theorem 6.1.7).

There exists a direct simple proof of this fact. Let f be a continuous real-valued function defined on σA. For any $z \in \rho A$ fix $x(z) \in (z \cdot H) \cap \sigma A$. By virtue of Lemma 5.3, $z \cdot H = \beta((z \cdot H) \cap \sigma A)$. Thus there exists a unique continuous extension of f onto $z \cdot H$. This extension we denote by g. We affirm that the function g is continuous.

Let F be a compact G_δ-subset of the space ρA. Then the set $Y = F \cap \sigma A$ is G_δ-dense in F. Since F is Dugundji compact, $F = \beta Y$ and the restriction of the function on F is continuous. Since ρA is a paracompact p-space and the restriction of g on compact G_δ-subsets is continuous, the function g is continuous on A. □

Let $\beta\psi : \beta\rho A \longrightarrow \beta B^\star$ be the continuous extension of the mapping ψ on the Stone-Čech compactifications of the spaces ρA and $B^\star$. Since ψ is a perfect mapping $\rho A = \beta\psi^{-1}(B^\star)$. Therefore $\sigma A = A \cup (\beta\rho A \setminus \beta\psi^{-1}(B))$ and $\beta\psi(\sigma A) = \beta B^\star$.

We put $\pi = \beta\psi|\sigma A$ and $\pi B = \beta B^\star$.

Property 5.6. *The mapping $\beta\psi$ is open and perfect and the mapping $\pi : \sigma A \longrightarrow \pi B$ is open.*

Proof. Follows from [6]. □

Property 5.7. *σA is a pseudocompact thin extension of A.*

Proof. Since $\sigma A = A \cup (\beta\rho A \setminus \beta\psi^{-1}(B)$ and $\beta\psi(\sigma A) = \beta B^\star$ the extension σA is thin. Remark 5.1 completes the proof. □

Property 5.8. *If $\mathcal{P} \leq S$, then A is an M-space, σA is a countably compact space and the mappings $\varphi : A \longrightarrow B$ and $\pi : \sigma A \longrightarrow \pi B$ are quasi-perfect.*

Proof. Let $U \supseteq P$ be an open subset of A. Assume that $W_n \setminus U \neq \varnothing$ for any $n \in \mathbb{N}$. Then the sequence $\{x_n \in W_n \setminus U : n \in \mathbb{N}\}$ exists and has an accumulation point in $x_0 \in A$. By construction, $x_0 \in A \setminus U$ and $x_0 \notin W_m$ for some $m \in \mathbb{N}$. The accumulation points of the sequence $\{x_n \in W_n : n \in \mathbb{N}\}$ are from P, a contradiction. □

Property 5.9. *If $\mathcal{P} \leq k$, then A is a paracompact p-space.*

Proof. In this case φ is a perfect mapping of A onto a metric space B. □

Property 5.10. *The space A is p-embedded in σA.*

Proof. The mapping $\pi : \sigma A \longrightarrow \pi B$ is continuous, $\pi^{-1}(B) = A$ and B, as a metrizable space, is p-embedded in πB. □

Property 5.11. *Let X be a dense subspace of the space A. If X contains a complete dense-Π-sieve, then A is a G_δ-subspace of the extension σA.*

Proof. Let X contains a complete dense-Π-sieve. Since the mapping $\varphi : A \longrightarrow B$ is open, continuous and the fibers $\varphi^{-1}(y)$, $y \in B$, are pseudocompact, the space B contains a dense Čech complete subspace Y. The set $Z = \varphi^{-1}(Y)$ is a dense G_δ-subset of σA and $Z \subseteq A$. Assume that $A \neq \sigma A$. Fix $c \in \sigma A \setminus A$. The subspace cZ is dense in σA and has the Baire property as a space homeomorphic with the space Z. By construction, $\sigma A \setminus Z$ is of the first category and contains cZ, a contradiction. Thus $\sigma A = A$. □

Property 5.12. *If A contains a dense Čech complete subspace, then A is a paracompact Čech complete space. In particular, A is a G_δ-subspace of the compactification $\sigma A = \beta A$.*

Proof. Follows from Property 5.11. This fact was proved in [2]. □

For the property S in similar way we obtain.

Property 5.13. *If A contains a complete dense-S-sieve, then A is a complete M-space and a G_δ-subspace of the extension σA.*

Remark 5.4. If $\mathcal{P} \in \{\Pi, S, k\}$, then σA is a $\mathcal{P}$-extension.

6. $(\beta, G_{\mathcal{P}})$-unfavorableness of paratopological groups

The main result of this section is the following theorem.

Theorem 6.1. *Let A be a paratopological group and $\mathcal{P} \leq \Pi$ be a stable property of sequences of open subsets of the subspaces of the space A. The following assertions are equivalent:*

1. The space A is a topological group with the Baire property and A contains a sequence of open subsets with the property $\mathcal{P}$.

2. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense Baire subspace X which is p-embedded in some pseudocompact extension.

3. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense Baire subspace X with a dense-Π*-sieve.*

4. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense subspace X which is $(\beta, G_{\mathcal{P}})$-unfavorable.

5. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense subspace X which is (β, G_{BM})-unfavorable and $p^{\star\star}$-embedded in some pseudocompact extension.

6. The space A is a topological group with the Baire property, contains a sequence of open subsets with the property $\mathcal{P}$ and is p-embedded in some pseudocompact extension.

Proof. Let A be a topological group and A contains a sequence of open subsets with the property $\mathcal{P}$. From Property 5.10 it follows that A is p-embedded in the pseudocompact extension σA. In particular, A has a dense-$\mathcal{P}i$-sieve. The implications $1 \to 5$ and $1 \to 4$ are proved. The implications $1 \to 6 \to 2$, $1 \to 2$ and $1 \to 3$ are obvious. The implication $5 \to 2$ is follows from Theorem 4.8. The other implications are proved in [4]. □

Corollary 6.2. *Let A be a paratopological group. The following assertions are equivalent:*

1. The space A is a topological group with the Baire property, A contains a sequence of open subsets with the property Π *and is p-embedded in some pseudocompact extension.*

2. The space A contains a dense Baire subspace X which is p-embedded in some pseudocompact extension.

3. The space A contains a dense Baire subspace X with a dense-Π*-sieve.*

4. The space A contains a dense subspace X which is (β, G_{Π})-unfavorable.

5. The space A contains a dense subspace X which is (β, G_{BM})-unfavorable and $p^{\star\star}$-embedded in some pseudocompact extension.

6. The space A has the Baire property and is p-embedded in some pseudocompact extension.

7. $(\beta, G_{\mathcal{P}})$-UNFAVORABLENESS OF SEMITOPOLOGICAL GROUPS

The main result of this section is the following theorem.

Theorem 7.1. *Let A be a semitopological group and $\mathcal{P} \leq S$ be a stable property of sequences of open subsets of the subspaces of the space A. The following assertions are equivalent:*

1. The space A is a topological group with the Baire property and A contains a sequence of open subsets with the property $\mathcal{P}$.

2. The space A is a topological group with the Baire property, contains a sequence of open subsets with the property $\mathcal{P}$ and the extension σA is countably compact.

3. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense Baire subspace X which is p-embedded in some countably compact space.

4. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense Baire subspace X with a dense-Π-sieve.

5. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense subspace X which is $(\beta, G_{\mathcal{P}})$-unfavorable.

6. The space A contains a sequence of open subsets with the property $\mathcal{P}$ and a dense subspace X which is (β, G_{BM})-unfavorable and $p^{\star\star}$-embedded in some countably compact space.

Proof. The implication $5 \to 1$ is proved in [24, 4]. Theorem 6.1 completes the proof. □

Corollary 7.2. *Let A be a semitopological group. The following assertions are equivalent:*

1. The space A is a topological group with the Baire property and A contains a sequence of open subsets with the property S.

2. The space A has the Baire property and is p-embedded in some countably compact space.

3. The space A contains a sequence of open subsets with the property S and a dense Baire subspace X which is p-embedded in some countably compact space.

4. The space A contains a sequence of open subsets with the property S and a dense subspace X which is (β, G_S)-unfavorable.

5. The space A contains a sequence of open subsets with the property S a dense subspace X which is (β, G_{BM})-unfavorable and $p^{\star\star}$-embedded in some countably compact space.

Corollary 7.3. *Let A be a semitopological group. The following assertions are equivalent:*

1. The space A is a topological group with the Baire property and a paracompact p-space.

2. The space A is a topological group with the Baire property and σA is a compactification of A.

3. The space A contains a dense Baire subspace X which is a $p^{\star\star}$-space.

4. The space A contains a dense Baire subspace X with a dense-k-sieve.

5. The space A contains a dense subspace X which is (β, G_k)-unfavorable.

6. The space A contains a dense subspace X which is (β, G_{BM})-unfavorable and $p^{\star\star}$-embedded in some compact space.

8. On Hausdorff locally countably compact spaces

B. Bokalo and I. Guran [9] have established that a Hausdorff sequentially compact cancellative semigroup is a topological group.

Now it is natural to formulate the following questions.

Problem 8.1. *Let G be a Hausdorff feebly compact paratopological group of pointwise countable type. Is it true that G is a topological group?*

Problem 8.2. *Let G be a regular semitopological group of pointwise countable type and G be a dense G_δ-subspace of some feebly compact space. Is it true that G is a topological group?*

Problem 8.3. *Let G be a regular semitopological group, G be a dense G_δ-subspace of some feebly compact space and G be a q-space. Is it true that G is a topological group?*

In [30] O.Ravsky has constructed an MA-example of Hausdorff countably compact paratopological group which is not a topological group.

Lemma 8.4. *Let X be a Hausdorff locally countably compact space of pointwise countable type. Then X is a regular space.*

Proof. Let F be a closed subset of X and $x \in X \setminus F$. There exists a compact set K of countable character in X such that $x \in K$. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of open subsets of X such that:

- $K \subseteq U_{n+1} \subseteq U_n$ for any $n \in \mathbb{N}$;
- for any open set $U \supseteq K$ there exists $n \in \mathbb{N}$ such that $U_n \subseteq U$.

We put $F_1 = F \cap K$. Since F_1 is a compact set and $x \in X \setminus F_1$, there exist two open subsets V_1 and W_1 of X such that $x \in V_1$, $F_1 \subseteq W_1$, $V_1 \cap W_1 = \varnothing$ and the set $Y = cl_X V_1$ is countably compact.

Let $F_2 = Y \cap (F \setminus W_1)$. The set F_2 is closed in X. Since X is a Hausdorff space and K is a compact subset, $K = \cap\{cl_X U_n : n \in \mathbb{N}\}$. Since $\cap\{F_2 \cap cl_X U_n : n \in \mathbb{N}\} = \varnothing$ and Y is a countably compact subset, there exists $n \in \mathbb{N}$ such that $F_2 \cap cl_X U_n = \varnothing$. Now we put $V = V_1 \cap U_n$ and $W = W_1 \cup (X \setminus cl_X U_n) \cup (X \setminus Y)$. Then $x \in V$, $F \subseteq W$ and $V \cap W = \varnothing$. □

From Lemma 8.4 and Corollary 7.2 it follows

Corollary 8.5. *Let A be a Hausdorff locally countably compact semitopological group of pointwise countable type. Then A is a paracompact locally compact topological group.*

9. Semitopological n-groups

Let $n \geq 2$. An n-ary group or an n-group is a family $(A, \{m, l, k\})$, where A is a non-empty set and $m, l, r : A^n \longrightarrow A$ are three n-ary operations on A with the properties:

1mg) $m(m(x_1, ..., x_n), x_{n+1}, ..., x_{2n-1}) = m(x_1, ..., x_i, m(x_{i+1}, ..., x_{i+n}), x_{i+n+1}, ..., x_{2n-1})$ for all $i < 2n - 1$ and $x_1, ..., x_{2n-1} \in A$;

2mg) $m((x_1, ..., x_{n-1}, r(x_1, ..., x_n)) = x_n$ and $m(l(x_1, ..., x_n), x_2, ..., x_n) = x_1$ for all $x_1, ..., x_n \in A$.

Definition 9.1. Let $n \geq 2$. An n-group $(A, \{m, l, r\})$ with a given topology on A is called:

(TG) a topological n-group if the mappings $m, l, r : A^n \longrightarrow A$ are continuous;

(PTG) a paratopological n-group if:

- for any $c = (c_1, ..., c_n) \in A^n$ the mapping $\mu_c : A^2 \longrightarrow A$, where $\mu_c(x, y) = m(x, c_2, ..., c_{n-1}, y)$ for each $(x, y) \in A^2$, is continuous;

- for any $c = (c_1, ..., c_n) \in A^n$ the mapping $\lambda_c : A \longrightarrow A$, where $\lambda_c(x) = m(c_1, c_2, ..., c_{n-2}, x, c_n)$ for each $x \in A$, is continuous;

(STG)) a semitopological n-group if:

- for any $c = (c_1, ..., c_n) \in A^n$ the mapping $\mu_c : A^2 \longrightarrow A$, where $\mu_c(x, y) = m(x, c_2, ..., c_{n-1}, y)$ for each $(x, y) \in A^2$, is separately continuous;
- for any $c = (c_1, ..., c_n) \in A^n$ the mapping $\lambda_c : A \longrightarrow A$, where $\lambda_c(x) = m(c_1, c_2, ..., c_{n-2}, x, c_n)$ for each $x \in A$, is continuous.

In [19] G.Čupona has initiated the study of topological n-groups and has raised some questions about their properties. Distinct definitions of topological n-groups were examined in [35].

Our aim is to prove that the results from the above sections are true for n-groups.

Let $n \geq 2$ and $(A, \{m, l, r\})$ be an n-group with a given topology. If $n = 2$, then $(A, \{m, l, r\})$ is a group.

Assume that $n \geq 3$.

By virtue of Hosszu-Gluskin's theorem [35], there exists $c = (c_1, ..., c_n) \in A^n$ such that:

(P1HG). Relatively to the binary operation $x \cdot y = m(x, c_2, ..., c_{n-1}, y)$ the pair $(A, \cdot)$ is a group.

(P2HG). The mapping $\lambda_c : A \longrightarrow A$ is an automorphism of the group $(A, \cdot)$.

(P3HG). If $b = m(c_1, c_1, ..., c_1, c_1)$, then $\lambda_c(b) = b$ and $\lambda_c^{(n-1)}(x) \cdot b = b \cdot x$ for any $x \in A$.

(P4HG). $m(x_1, x_2, ..., x_{n-1}, x_n) = x_1 \cdot \lambda_c(x_2) \cdot ... \cdot \lambda_c^{(n-2)}(x_{n-1}) \cdot \lambda_c^{(n-1)}(x_n) \cdot b$ for all $x_1, x_2, ..., x_{n-1}, x_n \in A$.

These four assertions imply the following properties.

Property 9.1. $(A, \{m, l, r\})$ is a topological group if and only if $(A, \cdot)$ is a topological group and the automorphism λ_c is continuous.

Property 9.2. $(A, \{m, l, r\})$ is a paratopological group if and only if $(A, \cdot)$ is a paratopological group and the automorphism λ_c is continuous.

Property 9.3. $(A, \{m, l, r\})$ is a semitopological group if and only if $(A, \cdot)$ is a semitopological group and the automorphism λ_c is continuous.

Corollary 9.2. *The results about groups from the above sections are true for n-groups.*

10. Remarks. Open problems

By virtue of Corollary 2.5 from [8] and Corollaries 7.2 and 7.3 it follows:

Corollary 10.1. *Let A be a semitopological group of countable π-character. Then:*

1. If A contains a dense subspace with a complete dense-Π-sieve, then A is a completely metrizable topological group.

2. If A contains a dense Baire subspace with a dense-Π-sieve, then A is a metrizable group.

Now we mention some open problems.

Problem 10.2. *Let A be a semitopological group, A be a sequential (or Fréchet-Urysohn, or k-space) space and contain a complete dense-Π-sieve. Is it true that A is a topological group?*

Problem 10.3. *Let A be a semitopological group, A be a space of countable tightness and A be a dense G_δ-subspace of some feebly compact space. Is it true that A is a topological group?*

By virtue of Corollary 10.1, in the class of first countable spaces the answers to Problems 10.2 and 10.3 are positive.

On an arbitrary product of Čech complete spaces there exists a Markov winning strategy of the player α in the Banach-Mazur game (see [18]). If $\{X_\mu : \mu \in A\}$ is a family of infinite spaces and the set $\{\mu \in A : X_\mu \textit{ is not pseudocompact}\}$ is an uncountable set, then any strategy of the player β on $X = \Pi\{X_\mu : \mu \in A\}$ is winning in the G_Π-game on $X = \Pi\{X_\mu : \mu \in A\}$.

Problem 10.4. *Let A be a paratopological group (or a semitopological group) and the space A be homeomorphic to Cartesian product of an uncountable family of Čech complete spaces. Is it true that A is a topological group?*

Problem 10.5. *Let A be a paratopological group (or a semitopological group) and the space A be homeomorphic to the Cartesian product of an uncountable family of Čech complete Lindelöf spaces. Is it true that A is a topological group?*

Problem 10.6. *Let A be a paratopological group (or a semitopological group) and the space A be homeomorphic to the Cartesian product of an uncountable family of infinite discrete spaces. Is it true that A is a topological group?*

Problem 10.7. *Let A be a paratopological group (or a semitopological group), $\mathbb{R}$ be the space of reals, m be an uncountable cardinal number and the space A be homeomorphic to the space $\mathbb{R}^m$. Is it true that A is a topological group?*

Problem 10.8. *Let A be a paratopological group (or a semitopological group), $\mathbb{N}$ be the discrete space of integers, m be an uncountable cardinal number and the space A be homeomorphic to the space $\mathbb{N}^m$. Is it true that A is a topological group?*

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