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A NOTE ON THE BESSEL POLYNOMIALS

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Abstract. Starting from a recently considered integral representation for the Bessel polynomials, the authors derive a Rodrigues formula, a relationship with the Laguerre polynomials, and a number of other interesting properties for the Bessel polynomials. Relevant connections of each of the results, which are obtained in this paper, with those considered in earlier works are also mentioned.

As long ago as 1949, Krall and Frink (see [5] and [6]; see also the references cited therein) initiated a systematic investigation of the Bessel polynomials. In their terminology, the *generalized Bessel polynomials* $y_n(x;\alpha,\beta)$ are defined by

$$\begin{aligned} y_n(x;\alpha,\beta) :&= \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k \\ &= {}_2F_0\left(-n, \alpha+n-1; —; -\frac{x}{\beta}\right) \end{aligned} \tag{1}$$

and the *simple Bessel polynomials* $y_n(x)$ are defined by

$$\begin{aligned} y_n(x) :&= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} k! \left(\frac{x}{2}\right)^k = y_n(x;2,2) \\ &= {}_2F_0\left(-n, n+1; —; -\frac{x}{2}\right). \end{aligned} \tag{2}$$

An immediate consequence of the definition (1) is the integral representation:

$$y_n(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha+n-1)} \int_0^\infty e^{-t} \left(1+\frac{xt}{\beta}\right)^n t^{\alpha+n-2}\, dt \tag{3}$$

$$(\Re(\alpha) > 1),$$

which, for $\alpha = \beta = 2$, was employed recently by Dattoli *et al.* [3, p. 321] with a view to deriving the generating function:

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$$\sum_{n=0}^{\infty} y_n(x)(-\tau)^n = \sqrt{\frac{\pi}{2x\tau}} \exp\left(\frac{(1+\tau)^2}{2x\tau}\right) \operatorname{erfc}\left(\frac{1+\tau}{\sqrt{2x\tau}}\right) \tag{4}$$

$$(0<\tau<1;\ 0<x<1).$$

The generating function (4) would follow also upon setting

$$\alpha=\beta=2 \quad \text{and} \quad t=-\tau$$

in the known result [6, p. 139, Eq. 2.6 (11)]:

$$(1-t)^{1-\alpha}\,{}_2F_0\left[\frac{1}{2}(\alpha-1),\frac{1}{2}\alpha;—;\frac{4xt}{\beta(1-t)^2}\right] \cong \sum_{n=0}^{\infty}\frac{(\alpha-1)_n}{n!}\,y_n(x;\alpha,\beta)\,t^n, \tag{5}$$

since [6, p. 40, Eq. 1.3 (28)]

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-z^2\right)\Psi\left(\frac{1}{2},\frac{1}{2};z^2\right) \tag{6}$$

and [6, p. 38, Eq. 1.3 (16)]

$${}_2F_0\left(\lambda,\mu;—;-\frac{1}{z}\right) = z^{\lambda}\,\Psi(\lambda,\lambda-\mu+1;z) \tag{7}$$

in terms of the Tricomi function $\Psi(a,c;z)$.

We can make use of the integral on the right-hand side of the equation (3) in order to get a further representation of operational nature for $y_n(x;\alpha,\beta)$. Indeed, by introducing a suitable parameter ξ, we can recast the integral on the right-hand side of (3) as follows:

$$y_n(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha+n-1)}\int_0^{\infty}\exp\left[-t\left(1-\frac{x}{\beta}\frac{\partial}{\partial\xi}\right)\right]t^{n+\alpha-2}\,\xi^n\,dt\bigg|_{\xi=1}, \tag{8}$$

whose formal integration yields the following operational representation:

$$y_n(x;\alpha,\beta) = \left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial\xi}}\right)^{\alpha+n-1}\xi^n\bigg|_{\xi=1}, \tag{9}$$

which can be looked upon as a Rodrigues formula for the Bessel polynomials $y_n(x;\alpha,\beta)$.

If we replace α in (9) by $\alpha-2n+1$ and use the operational identity [4]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left(1+\kappa \frac{d}{dx}\right)^n x^n = \exp\left(\left\{\log (1-\kappa t)^{-1/\kappa}\right\} \left(1+\kappa \frac{d}{dx}\right) x\right), \tag{10}$$

we find that

$$\sum_{n=0}^{\infty} y_n (x; \alpha - 2n + 1, \beta) \frac{t^n}{n!} = \left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial \xi}}\right)^{\alpha} \cdot \exp\left[\left\{\log\left(1+\frac{xt}{\beta}\right)^{\beta/x}\right\}\left(\xi - \frac{x}{\beta} - \frac{x}{\beta}\xi\frac{\partial}{\partial \xi}\right)\right]\Bigg|_{\xi=1}. \tag{11}$$

By introducing the operators $\hat{A}$ and $\hat{B}$ defined by

$$\hat{A} = \xi \qquad \text{and} \qquad \hat{B} = -\frac{x}{\beta}\xi\frac{d}{d\xi}, \tag{12}$$

and by applying the decomposition rule [1]:

$$e^{\hat{A}+\hat{B}} = \exp\left(\frac{e^k - 1}{k}\hat{A}\right) e^{\hat{B}} \qquad \text{with} \qquad \left[\hat{A}, \hat{B}\right] := \hat{A}\hat{B} - \hat{B}\hat{A} = -k\hat{A}, \tag{13}$$

we end up with the following generating function:

$$\begin{aligned}\sum_{n=0}^{\infty} y_n (x; \alpha - 2n + 1, \beta) \frac{t^n}{n!} &= \left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial \xi}}\right)^{\alpha} \frac{1}{1+\frac{xt}{\beta}} \exp\left(\frac{t\xi}{1+\frac{xt}{\beta}}\right)\Bigg|_{\xi=1} \\ &= \left(1+\frac{xt}{\beta}\right)^{\alpha-1} \exp\left(\frac{t}{1+\frac{xt}{\beta}}\right),\end{aligned} \tag{14}$$

which is also an immediate consequence of the following much more general known result due to Srivastava (see, *e.g.*, [6, p. 92, Problem 18]) for $\sigma = -2$ and $\alpha \longmapsto \alpha + 1$:

$$\sum_{n=0}^{\infty} y_n (x; \alpha + \sigma n, \beta) \frac{t^n}{n!} = \frac{(1-u)^{\alpha-1}}{1+(\sigma+1)u} \exp\left(-\frac{\beta u}{x(1-u)}\right), \tag{15}$$

where

$$u = u(t) = -\frac{xt}{\beta}(1-u)^{\sigma+2} \qquad \text{and} \qquad u(0) = 0. \tag{16}$$

By recalling that the two-variable Laguerre polynomials are defined through the generating function [2]:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,w)\, t^n = (1-wt)^{-\alpha-1} \exp\left(-\frac{xt}{1-wt}\right) \qquad \left(|t| < |w|^{-1}\right), \tag{17}$$

by comparison with the equation (14), we obtain the relationship:

$$y_n(x;\alpha-2n+1,\beta) = n!\, L_n^{(-\alpha)}\left(-1,-\frac{x}{\beta}\right). \tag{18}$$

Furthermore, since the polynomials $L_n(x,w)$ are linked to the ordinary Laguerre polynomials $L_n(x) := L_n^{(0)}(x)$ by means of relationship:

$$L_n(x,w) = w^n\, L_n\left(\frac{x}{w}\right), \tag{19}$$

we find that

$$y_n(x;\alpha-2n+1,\beta) = n!\left(-\frac{x}{\beta}\right)^n L_n^{(-\alpha)}\left(\frac{\beta}{x}\right), \tag{20}$$

which, for $\alpha \longmapsto \alpha+2n-1$, leads us at once to the following well-known relationship [6, p. 92, Problem 18]:

$$y_n(x;\alpha,\beta) = n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right) \tag{21}$$

with the associated Laguerre polynomials $L_n^{(\alpha)}(x)$.

In view of the relationship (20) and the integral representation (8), we can introduce the following operational definition of the ordinary Laguerre polynomials:

$$L_n(\zeta) = \frac{(-1)^n}{n!}\left(1-\frac{1}{\zeta}\frac{\partial}{\partial\xi}\right)^n (\zeta\xi)^n\bigg|_{\xi=1}. \tag{22}$$

And, for the more general definition for the associated forms, we have

$$L_n^{(m)}(\zeta) = \frac{(-1)^n}{n!}\left(1-\frac{1}{\zeta}\frac{\partial}{\partial\xi}\right)^{n+m} (\zeta\xi)^n\bigg|_{\xi=1}, \tag{23}$$

where m is assumed to be an integer. The extension to any real or complex m will be considered elsewhere. These last relations (22) and (23) are similar, but not identical, to analogous operational formulas derived in [2] in connection with the monomiality principle.

In our operational representation (9), we have imposed the condition that, after the evaluation of the various derivatives, the parameter ξ should be set equal to 1. If we waive such a restriction, we can generalize the Bessel polynomials as follows:

$$y_n(x;\alpha,\beta;\xi) = \left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial\xi}}\right)^{\alpha+n-1} \xi^n = \xi^n\, y_n\left(\frac{x}{\xi};\alpha,\beta\right). \tag{24}$$

This extension is useful to derive generating functions of the following known type [5, p. 105, Eq. (1.23)]:

$$\begin{aligned}\sum_{m=0}^{\infty} y_n(x;\alpha-m,\beta)\frac{t^m}{m!} &= e^t\, y_n\left(x;\alpha,\beta;1-\frac{xt}{\beta}\right)\\ &= \left(1-\frac{xt}{\beta}\right)^n e^t\, y_n\left(\frac{\beta x}{\beta - xt};\alpha,\beta\right),\end{aligned} \tag{25}$$

which is a fairly straightforward consequence of the integral representation (8).

We can now introduce the new forms of the Bessel polynomials, which offer new insight as well as the possibility of extending the families of generating functions discussed so far.

We begin by defining

$$y_n(x;\alpha,\beta;\xi,\zeta) = \left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial\xi}}\right)^{\alpha+n-1} H_n(\xi,\zeta), \tag{26}$$

where

$$H_n(\xi,\zeta) = n!\sum_{r=0}^{[n/2]}\frac{\xi^{n-2r}\,\zeta^r}{(n-2r)!\,r!} = \left(i\sqrt{\zeta}\right)^n H_n\left(\frac{\xi}{2i\sqrt{\zeta}}\right) \qquad \left(i := \sqrt{-1}\right) \tag{27}$$

in terms of the familiar Hermite polynomials $H_n(x)$. Now, on account of the operational definition [2]:

$$H_n(\xi,\zeta) = \exp\left(\zeta\frac{\partial^2}{\partial\xi^2}\right)\xi^n, \tag{28}$$

(26) yields

$$y_n(x;\alpha,\beta;\xi,\zeta) = \exp\left(\zeta\frac{\partial^2}{\partial\xi^2}\right) y_n(x;\alpha,\beta;\xi). \tag{29}$$

According to this last result (29), we can easily find that

$$\begin{aligned}\sum_{m=0}^{\infty} y_n(x;\alpha-2m,\beta)\frac{t^m}{m!} &= \exp\left[t\left(1-\frac{2x}{\beta}\frac{\partial}{\partial\xi}+\frac{x^2}{\beta^2}\frac{\partial^2}{\partial\xi^2}\right)\right] y_n(x;\alpha,\beta;\xi)\Big|_{\xi=1}\\ &= e^t\, y_n\left(x;\alpha,\beta;1-\frac{2xt}{\beta},\frac{x^2t}{\beta^2}\right).\end{aligned} \tag{30}$$

See also the above observation in connection with (14).

The use of higher-order Hermite polynomials [2] allows the extension of this last identity (30) to the following case:

$$\sum_{m=0}^{\infty} y_n\left(x;\alpha-sm,\beta\right)\frac{t^m}{m!}$$
$$= e^t y_n\left(x;\alpha,\beta;1-\binom{s}{1}\frac{xt}{\beta},\binom{s}{2}\left(-\frac{x}{\beta}\right)^2 t,\ldots,\binom{s}{r}\left(-\frac{x}{\beta}\right)^r t,\ldots,\binom{s}{s}\left(-\frac{x}{\beta}\right)^s t\right), \tag{31}$$

where

$$y_n\left(x;\alpha,\beta;\xi_1,\ldots,\xi_s\right)=\left(\frac{1}{1-\frac{x}{\beta}\frac{\partial}{\partial\xi}}\right)^{\alpha+n-1} H_n^{(s)}\left(\xi_1,\ldots,\xi_s\right), \tag{32}$$

and

$$H_n^{(s)}\left(\xi_1,\ldots,\xi_s\right)=\sum_{r=0}^{[n/s]}\frac{n!}{r!\,(n-sr)!}H_{n-sr}^{(s-1)}\left(\xi_1,\ldots,\xi_{s-1}\right)\xi_s^r,$$

with the multivariable polynomials $H_n^{(s)}\left(\xi_1,\ldots,\xi_s\right)$ defined through the operational relation:

$$H_n^{(s)}\left(\xi_1,\ldots,\xi_s\right)=\exp\left(\sum_{r=2}^{s}\xi_r\frac{\partial^r}{\partial\xi_1^r}\right)\xi_1^n. \tag{33}$$

The properties of this new family of Bessel polynomials defined by (32) are fairly interesting, too. Indeed, by noting the operational relationship (29), we find from (13) that

$$\sum_{n=0}^{\infty} y_n\left(x;\alpha-2n+1,\beta,\xi,\zeta\right)\frac{t^n}{n!}=\exp\left(\zeta\frac{\partial^2}{\partial\xi^2}\right)\left(1+\frac{xt}{\beta}\right)^{\alpha-1}\exp\left(\frac{t\xi}{1+\frac{xt}{\beta}}\right)$$
$$=\left(1+\frac{xt}{\beta}\right)^{\alpha-1}\exp\left(\frac{t\left(\xi+2\zeta\frac{\partial}{\partial\xi}\right)}{1+\frac{xt}{\beta}}\right), \tag{34}$$

which, on account of the Weyl identity:

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}\,e^{\hat{B}}\,e^{k/2}\quad\text{with}\quad\left[\hat{A},\hat{B}\right]:=\hat{A}\hat{B}-\hat{B}\hat{A}=-k, \tag{35}$$

yields the generating function:

$$\sum_{n=0}^{\infty} y_n(x; \alpha - 2n + 1; \beta, \xi, \zeta) \frac{t^n}{n!} = \left(1 + \frac{xt}{\beta}\right)^{\alpha - 1} \exp\left[\frac{t\xi}{1 + \frac{xt}{\beta}} + \left(\frac{t}{1 + \frac{xt}{\beta}}\right)^2 \zeta\right]. \tag{36}$$

In conclusion, the results of this note show that significant progress in the theory of Bessel polynomials can be made by employing a formalism of operational nature. In a forthcoming investigation, we will discuss further results and applications.

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