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ON AN INTEGRAL TRANSFORM INVOLVING BESSEL FUNCTIONS

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Abstract

This paper deals with a new integral transform, involving a combination of Bessel functions as a kernel. The inversion formula is established and some properties are given. This transform can be used to solve some mixed boundary value problems. We consider here a problem of heat conductions in an infinite and semi-infinite cylinder ($r = a$, $r = b$, $b > a$) with radiation-type boundary conditions.

1. Introduction

Let $f(t)$ be a given function defined on an interval $[a, b]$, that belongs to a certain class of functions. An integral transform of $f(t)$ is a mapping of the form,

$$T[f(t); s] = \bar{f}(s) = \int_a^b K(s,t)f(t)dt,$$

provided that the integral exists. $K(s, t)$ is a prescribed function, called the kernel of the transform [2,5,6,11]. Among the well known transforms are the Laplace, Fourier, Hankel, Stieltjes and Mellin transforms. The most versatile of these, the Laplace transform has been widely used to solve differential equations, and particularly problems related to heat transfer and electrical circuits. On the other hand for problems in which there is an axial symmetry, the Hankel transforms are found to be most appropriate. Mellin transform being closely related to the Fourier transform, has its own peculiar uses, as for deriving expansion and solving problems with wedge shape boundaries. In general, the use of an integral transform often reduces a partial differential equation in n independent variables to $(n-1)$ variables, that provides a simplification of the problem.

The success of the use of integral transforms to solve boundary value problems and to exclude a variable with range $(0,\infty)$ or $(-\infty,\infty)$ led investigators to consider finite integral transforms. Doetsch considered finite Fourier transforms, and Sneddon [11] extended the idea of Bessel function kernel, called 'Finite Hankel Transforms'. Using the Sturm-Liouville theory [3,], a number of integral transforms can be implemented, according to the prescribed boundary conditions.

Recently Khajah [9] has considered a modified Hankel transform in the form,

$$J_\mu[f(z); s, \lambda] = \int_0^b z^\lambda f(z) J_\mu(zs)dz$$

where $f(z)$ satisfies Dirichlet's conditions on the interval $[a,b]$. He has derived the inversion formula, Parseval-type identities, transform of derivatives, as well as transforms of products of the form $z^\lambda f(z)$.

Using the Sturm-Liouville theory Kalla and Villalobos [7,8] have defined and studied an integral transform defined as,

$$T[f(x), a, b, \nu; \lambda_i] = \bar{f}_\nu(\lambda_i) = \int_a^b x f(x) C_\nu(\lambda_i x)dx,$$

where

$$\begin{aligned} C_\nu(\lambda_i x) = &\ \{Y_\nu(\lambda_i a) + B_\nu(\lambda_i b)\} J_\nu(\lambda_i x) \\ &- \{J_\nu(\lambda_i a) + A_\nu(\lambda_i b)\} Y_\nu(\lambda_i x) \end{aligned}$$

and

$$A_\nu(\lambda x) = J_\nu(\lambda x) + h\lambda J_\nu'(\lambda x)$$

$$B_\nu(\lambda x) = Y_\nu(\lambda x) + h\lambda Y_\nu'(\lambda x)$$

and λ_i are the positive roots of equation,

$$J_\nu(\lambda a) B_\nu(\lambda b) - Y_\nu(\lambda a) A_\nu(\lambda b) = 0$$

This transform has been used to solve a heat conduction problem in an infinite cylinder bounded by the surface $r = a$, $r = b$ $(b > a)$.

In this paper we define and study a new integral transform involving Bessel functions of first and second kind, by invoking the Sturm-Liouville theory. Inversion formula is established and some properties are mentioned. The transform has been used to solve a heat conduction problem in an infinite and a semi- infinite circular cylinder, bounded by surfaces $r = a$ and $r = b$ $(b > a)$, with radiation-type boundary conditions on both surfaces.

2. Definition and Inversion Formula:

Consider Bessel's differential equation

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0\,, \qquad x \in [a, b] \tag{1}$$

with homogeneous boundary conditions:

$$y(a) + h_1 y'(a) = y(b) + h_2 y'(b) = 0 \tag{2}$$

The general solution of (1) is given by:

$$y(x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x) \tag{3}$$

where c_1, c_2 are arbitrary constants, and $J_\nu(x)$, $Y_\nu(x)$ are the Bessel functions of first and second kind respectively. To obtain a solution of (1) that satisfies conditions (2), we have

$$c_1 \left[J_\nu(\lambda a) + h_1 \lambda J_\nu'(\lambda a) \right] + c_2 \left[Y_\nu(\lambda a) + h_1 \lambda Y_\nu'(\lambda a) \right] = 0 \tag{4}$$

$$c_1 \left[J_\nu(\lambda b) + h_2 \lambda J_\nu'(\lambda b) \right] + c_2 \left[Y_\nu(\lambda b) + h_2 \lambda Y_\nu'(\lambda b) \right] = 0 \tag{5}$$

from which we deduce

$$\frac{c_1}{c_2} = -\frac{Y_\nu(\lambda a) + h_1 \lambda Y_\nu'(\lambda a)}{J_\nu(\lambda a) + h_1 \lambda J_\nu'(\lambda a)} = -\frac{Y_\nu(\lambda b) + h_2 \lambda Y_\nu'(\lambda b)}{J_\nu(\lambda b) + h_2 \lambda J_\nu'(\lambda b)} \tag{6}$$

Let

$$A_\nu(\lambda x, h_k) = J_\nu(\lambda x) + h_k \lambda J_\nu'(\lambda x)\,, \qquad k = 1, 2$$

$$B_\nu(\lambda x, h_k) = Y_\nu(\lambda x) + h_k \lambda Y_\nu'(\lambda x)\,, \qquad k = 1, 2$$

Then, the function given by (3) is a solution of equation (1), subject to the conditions (2), if λ is a root of the transcendental equation,

$$B_\nu(\lambda a, h_1) A_\nu(\lambda b, h_2) - A_\nu(\lambda a, h_1) B_\nu(\lambda b, h_2) = 0 \tag{7}$$

Henceforth, we take λ_i $(i = 1, 2, \ldots)$ to be the positive roots of equation (7). Then, from (4–5), we have

$$y_i(x) = \frac{c_1}{B_\nu(\lambda_i a, h_1)} \left[J_\nu(\lambda_i x) B_\nu(\lambda_i a, h_1) - A_\nu(\lambda_i a, h_1) Y_\nu(\lambda_i x)\right] \tag{8}$$

$$= \frac{c_1}{B_\nu(\lambda_i b, h_2)} \left[J_\nu(\lambda_i x) B_\nu(\lambda_i b, h_2) - A_\nu(\lambda_i b, h_2) Y_\nu(\lambda_i x)\right] \tag{9}$$

If we define

$$\mathbf{Z}_i = B_\nu(\lambda_i a, h_1) + B_\nu(\lambda_i b, h_2)\,, \qquad \mathbf{W}_i = A_\nu(\lambda_i a, h_1) + A_\nu(\lambda_i b, h_2)$$

then the following functions are taken to be solutions of (1–2):

$$M_\nu(\lambda_i x) = \mathbf{Z}_i\, J_\nu(\lambda_i x) - \mathbf{W}_i\, Y_\nu(\lambda_i x) \tag{10}$$

By Sturm-Liouville theory [3], the functions of the system (10) are orthogonal on the interval $[a, b]$ with weight function x, that is

$$\int_a^b x M_\nu(\lambda_i x)\, M_\nu(\lambda_j x) dx = \begin{cases} 0, & i \neq j \\ \mathcal{M}_\nu(\lambda_i), & i = j \end{cases} \tag{11}$$

where $\mathcal{M}_\nu(\lambda_i) = \|\sqrt{x} M_\nu(\lambda_i x)\|_2^2$ — the weighted L^2 norm. If a function $f(x)$ and its first derivative are piecewise continuous on the interval $[a, b]$, then the relation

$$T[f(x), a, b, \nu; \lambda_i] = \bar{f}_\nu(\lambda_i) = \int_a^b x f(x) M_\nu(\lambda_i x) dx \tag{12}$$

defines a linear integral transform. To derive the inversion formula for this transform, given the series expansion,

$$f(x) = \sum_{i=1}^{\infty} a_i M_\nu(\lambda_i x) \tag{13}$$

we multiply (13) by $x M_\nu(\lambda_j x)$ and integrate both sides with respect to x to get the coefficients:

$$a_i = \frac{1}{\mathcal{M}_\nu(\lambda_i)} \int_a^b x\, f(x) M_\nu(\lambda_i x)\, dx = \frac{\bar{f}_\nu(\lambda_i)}{\mathcal{M}_\nu(\lambda_i)}\,, \qquad i = 1, 2, \ldots \tag{14}$$

and the inversion formula becomes

$$f(x) = \sum_{i=1}^{\infty} \frac{\bar{f}_\nu(\lambda_i)}{\mathcal{M}_\nu(\lambda_i)} M_\nu(\lambda_i x) \,. \tag{15}$$

Using some well known properties of Bessel functions [12] we can easily derive the following relation:

$$\begin{aligned} 2\,\mathcal{M}_\nu(\lambda_i) = &\mathbf{Z}_i^2 \left[b^2 P(\lambda_i, b, \nu) - a^2 P(\lambda_i, a, \nu)\right] \\ &- 2\mathbf{Z}_i \mathbf{W}_i \left[b^2 Q(\lambda_i, b, \nu) - a^2 Q(\lambda_i, a, \nu)\right] \\ &+ \mathbf{W}_i^2 \left[b^2 R(\lambda_i, b, \nu) - a^2 R(\lambda_i, a, \nu)\right] \end{aligned} \tag{16}$$

in which

$$P(\lambda_i, \mu, \nu) = J_\nu^2(\lambda_i \mu) - J_{\nu-1}(\lambda_i \mu) J_{\nu+1}(\lambda_i \mu)$$

$$R(\lambda_i, \mu, \nu) = Y_\nu^2(\lambda_i \mu) - Y_{\nu-1}(\lambda_i \mu) Y_{\nu+1}(\lambda_i \mu)$$

and

$$Q(\lambda_i, \mu, \nu) = J_\nu'(\lambda_i \mu) Y_{\nu-1}(\lambda_i \mu) - \frac{1}{\lambda_i \mu} J_{\nu-1}(\lambda_i \mu) Y_\nu(\lambda_i \mu) - J_{\nu-1}'(\lambda_i \mu) Y_\nu(\lambda_i \mu)$$

and μ stands for a or b. It is not difficult to verify some properties of the transform from definition (12). For example,

$$T[\alpha f(x) + \beta g(x), a, b, \nu; \lambda_i] = \alpha \bar{f}(\lambda_i) + \beta \bar{g}(\lambda_i) \tag{17}$$

$$T[f(px), a, b, \nu; \lambda_i] = \int_a^b x f(px) M_\nu(\lambda_i x) dx = \frac{1}{p^2} T[f(x), pa, pb, \nu; \lambda_i/p] \tag{18}$$

3. Transform of a Differential Operator:

We derive the transform of the following operator

$$Df(x) = \frac{d^2}{dx^2} f(x) + \frac{1}{x} \frac{d}{dx} f(x) - \frac{\nu^2}{x^2} f(x) \,, \qquad a \le x \le b \tag{19}$$

Let I be the transform of the first two terms of D, that is

$$\begin{aligned} I &= \int_a^b x \left[f''(x) + \frac{1}{x} f'(x)\right] M_\nu(\lambda_i x)\, dx \\ &= \int_a^b x f''(x) M_\nu(\lambda_i x) dx + \int_a^b f'(x) M_\nu(\lambda_i x) dx \end{aligned}$$

Integration by parts of the first integral leads to,

$$\int_a^b xf''(x)M_\nu(\lambda_i x)dx = xM_\nu(\lambda_i x)f'(x)\Big|_a^b - \int_a^b f'(x)\left[x\lambda_i M_\nu'(\lambda_i x)+M_\nu(\lambda_i x)\right]dx,$$

and hence,

$$I = xM_\nu(\lambda_i x)f'(x)\Big|_a^b - \lambda_i \int_a^b xf'(x)\, M_\nu'(\lambda_i x)\, dx$$

Integrating by parts once again leads to,

$$\begin{aligned} I &= x\left[f'(x)M_\nu(\lambda_i x) - \lambda_i f(x)M_\nu'(\lambda_i x)\right]\Big|_a^b \\ &\quad + \int_a^b x^{-1}\left[\lambda_i^2 x^2 M_\nu''(\lambda_i x) + \lambda_i x M_\nu'(\lambda_i x)\right] f(x)\, dx \end{aligned}$$

Since M_ν satisfies (1)we have

$$\lambda_i^2 x^2 M_\nu''(\lambda_i x) + \lambda_i x M_\nu'(\lambda_i x) = (\nu^2 - \lambda_i^2 x^2)\, M_\nu(\lambda_i x)$$

and

$$\int_a^b x^{-1}\left[\lambda_i^2 x^2 M_\nu''(\lambda_i x)+\lambda_i x M_\nu'(\lambda_i x)\right] f(x)\, dx = \int_a^b x\left[\frac{\nu^2}{x^2}-\lambda^2\right] f(x)M_\nu(\lambda_i x)dx$$

Furthermore, it follows from the boundary conditions (2) that

$$\lambda_i M_\nu'(\lambda_i a) = \frac{1}{h_1} M_\nu(\lambda_i a)\,, \qquad \lambda_i M_\nu'(\lambda_i b) = \frac{1}{h_2} M_\nu(\lambda_i b)$$

Hence

$$\begin{aligned} I &= \frac{b}{h_2} M_\nu(\lambda_i b)\left[f(b) + h_2 f'(b)\right] \\ &\quad - \frac{a}{h_1} M_\nu(\lambda_i a)\left[f(a) + h_1 f'(a)\right] - \lambda_i^2 \bar{f}(\lambda_i) + T\left[\frac{\nu^2}{x^2} f(x)\right] \end{aligned}$$

and the transform of the operator D in (19) becomes

$$\begin{aligned} T\left[Df(x)\right] = & \frac{b}{h_2} M_\nu(\lambda_i b)\left[f(b) + h_2 f'(b)\right] \\ & - \frac{a}{h_1} M_\nu(\lambda_i a)\left[f(a) + h_1 f'(a)\right] - \lambda_i^2 \bar{f}(\lambda_i) \end{aligned} \tag{20}$$

Transform of x^ν

From definition (12) we have

$$T[x^\nu, a, b, \nu; \lambda_i] = \int_a^b x^{\nu+1} M_\nu(\lambda_i x)\, dx$$

Using a result of [12], namely

$$\int x^{\rho+1} \mathcal{Z}_\rho(x) dx = x^{\rho+1} \mathcal{Z}_{\rho+1}(x)$$

where $\mathcal{Z}_\rho(x)$ is any of the Bessel functions, we obtain

$$T[x^\nu, a, b, \nu; \lambda_i] = \frac{1}{\lambda_i}\left[b^{\nu+1} M_{\nu+1}(\lambda_i b) - a^{\nu+1} M_{\nu+1}(\lambda_i a)\right]$$

Since

$$M_{\nu+1}(cz) = \frac{\nu}{cz} M_\nu(cz) - M_\nu'(cz)$$

the transform becomes,

$$\begin{aligned} & T[x^\nu, a, b, \nu; \lambda_i] = \\ & = \frac{b^{\nu+1}}{\lambda_i}\left[\frac{\nu}{\lambda_i b} M_\nu(\lambda_i b) - M_\nu'(\lambda_i b)\right] - \frac{a^{\nu+1}}{\lambda_i}\left[\frac{\nu}{\lambda_i a} M_\nu(\lambda_i a) - M_\nu'(\lambda_i a)\right] \end{aligned}$$

Then, from the boundary conditions (2) this reduces to,

$$T[x^\nu, a, b, \nu; \lambda_i] = \frac{b^{\nu+1}}{\lambda_i^2}\left[\frac{\nu}{b} + \frac{1}{h_2}\right] M_\nu(\lambda_i b) - \frac{a^{\nu+1}}{\lambda_i^2}\left[\frac{\nu}{a} + \frac{1}{h_1}\right] M_\nu(\lambda_i a) \tag{21}$$

In particular, the transform of a constant (where $\nu = 0$) is found to be

$$T[c, a, b, 0; \lambda_i] = \frac{c}{\lambda_i^2}\left[\frac{b}{h_2} M_0(\lambda_i b) - \frac{a}{h_1} M_0(\lambda_i a)\right] = c\,T[1, a, b, 0; \lambda_i] \tag{22}$$

4. Heat Conduction in An infinite Cylinder:

Consider a long hollow cylinder of inner radius a and outer radius b, with radiation type boundary conditions in both outer and inner surface, and a prescriped initial temperature. The differential equation of the phenomena is:

$$\frac{1}{K}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}, \tag{23}$$

where $a < r < b$, $t > 0$ and $U(r,t)$ denotes the temperature at any radial position r at time t; K is a constant that depends on the material of the cylinder. The initial and boundary conditions are as follows:

$$U(r,0) = I(r), \qquad a < r < b \tag{24}$$

$$U + h_1 \frac{\partial U}{\partial r}\bigg|_{r=a} = f(t), \quad U + h_2 \frac{\partial U}{\partial r}\bigg|_{r=b} = g(t), \qquad t > 0 \tag{25}$$

Taking $\nu = 0$, we consider the transform of U with respect to the radial variable, that is

$$\overline{U}(\lambda_i, t) = \int_a^b r\, U(r,t)\, M_0(\lambda_i r)\, dr \tag{26}$$

Referring to (20) and (23), we obtain

$$\frac{1}{K}\frac{\partial \overline{U}}{\partial t} = \frac{b}{h_2} M_0(\lambda_i b)\left[U + h_2 \frac{\partial U}{\partial r}\right]_{r=b} - \frac{a}{h_1} M_0(\lambda_i a)\left[U + h_1 \frac{\partial U}{\partial r}\right]_{r=a} - \lambda_i^2 \overline{U}$$

From the boundary conditions (25) we have this reduced to

$$\frac{1}{K}\frac{\partial \overline{U}}{\partial t} = \frac{b}{h_2} M_0(\lambda_i b)\, g(t) - \frac{a}{h_1} M_0(\lambda_i a)\, f(t) - \lambda_i^2 \overline{U}$$

and the following ODE is obtained

$$\frac{\partial \overline{U}}{\partial t} + K\lambda_i^2 \overline{U} = K\left[\frac{b}{h_2} M_0(\lambda_i b)\, g(t) - \frac{a}{h_1} M_0(\lambda_i a)\, f(t)\right] \tag{27}$$

whose solution is given by

$$\overline{U}(\lambda_i, t) = \exp(-K\lambda_i^2 t) \times$$

$$\times \left[K \int_0^t \exp(K\lambda_i^2 s)\left[\frac{b}{h_2} M_0(\lambda_i b)\, g(s) - \frac{a}{h_1} M_0(\lambda_i a)\, f(s)\right] ds + C\right]$$

Taking the transform of the initial condition (24), namely

$$\overline{U}(\lambda_i, 0) = \overline{I}(\lambda_i)$$

leads to $C = \overline{I}(\lambda_i)$, hence

$$\overline{U}_s(\lambda_i; t) = \exp(-K\lambda_i^2 t)$$

$$\Big[K \int_0^t \exp(-K\lambda s\big[\frac{b}{h_2} M_0(\lambda_i b)\, g(s) - \frac{a}{h_1} M_0(\lambda_i a)\, f(s)\big]\, ds + \overline{I}(\lambda_i)\Big] \quad (28)$$

The solution of (23) follows after applying the inversion formula to the above, thus

$$U(r,t) = \sum_{i=1}^{\infty} \frac{\overline{U}(\lambda_i, t)}{\mathcal{M}_\nu(\lambda_i)} M_0(\lambda_i r) \quad (29)$$

with the understanding that the summation is taken over all the positive roots of the equation:

$$B_0(\lambda a, h_1) A_0(\lambda b, h_2) - A_0(\lambda a, h_1) B_0(\lambda b, h_2) = 0 \quad (30)$$

Special cases:

(i) Let us consider the previous problem with the following initial and boundary conditions;

$$U(r,0) = 0\,, \qquad a < r < b \quad (31)$$

$$U + h_1 \frac{\partial U}{\partial r}\bigg|_{r=a} = U_0 \ \text{(const.)}, \quad U + h_2 \frac{\partial U}{\partial r}\bigg|_{r=b} = U_1 \ \text{(const.)}, \quad t > 0 \quad (32)$$

Then equation (28) will become

$$\overline{U}(\lambda_i; t) = \exp(-K\lambda_i^2 t)$$

$$\left[K \int_0^t \exp(K\lambda_i^2 x) \left\{\frac{b}{h_2} M_0(\lambda_i b) U_1 - \frac{a}{h_1} M_0(\lambda_i a) U_0\right\} dx\right] = \quad (33)$$

$$= \frac{1 - \exp(-K\lambda_i^2 t)}{\lambda_i^2} \left[\frac{bU_1}{h_2} M_0(\lambda_i b) - \frac{aU_0}{h_1} M_0(\lambda_i a)\right]$$

And according to (29) the solution will become

$$U(r,t)=\sum_{i=1}^{\infty}\frac{1-\exp(-K\lambda_i^2 t)}{\lambda_i^2\mathcal{M}_\nu(\lambda_i)}\times$$
$$\times\left[\frac{bU_1}{h_2}M_0(\lambda_i b)-\frac{aU_0}{h_1}M_0(\lambda_i a)\right]M_0(\lambda_i r),\qquad i=1,2,3,\ldots \tag{34}$$

where the summation is taken over all positive roots of (30).

(ii) If $h_1 \longrightarrow 0$ in (25), that is $U\,|_{r=a}=f(t)$, our result (29) reduces to a result of Kalla and Villalobos [8, p.41, eq.(20)].

5. Heat conduction in a semi-infinite cylinder:

Let us consider the problem of finding the temperature distribution $U(r,z,t)$ in a hollow semi-infinite cylinder with outer radius a and inner radius b. This problem is expressed by the differential equation,

$$\frac{1}{K}\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial z^2} \tag{35}$$

where $a<r<b$; $z,\ t>0$, and the initial/boundary conditions are taken to be:

$$U(r,z,0)=I(r,z) \tag{36}$$

$$U(r,0,t)=V(r,t)\,,\qquad \lim_{z\longrightarrow\infty}U(r,z,t)=0 \tag{37}$$

$$U+h_1\frac{\partial U}{\partial r}\bigg|_{r=a}=f(z,t)\,,\qquad U+h_2\frac{\partial U}{\partial r}\bigg|_{r=b}=g(z,t) \tag{38}$$

Consider the integral transform,

$$\overline{U}(\lambda_i,z,t)=\int_a^b r\,U(r,z,t)M_0(\lambda_i r)\,dr \tag{39}$$

and the Fourier sine transform

$$\overline{U}_s(\lambda_i,p,t)=\int_0^\infty \overline{U}(\lambda_i,z,t)\,\sin(p\,z)\,dz \tag{40}$$

Following a similar argument as in the previous section, the transformed equation of (35) becomes

$$\frac{1}{K}\frac{\partial\overline{U}}{\partial t}=\frac{b}{h_2}M_0(\lambda_i b)g(z,t)-\frac{a}{h_1}M_0(\lambda_i a)f(z,t)-\lambda_i^2\bar{U}+\frac{\partial^2\overline{U}}{\partial z^2}$$

whose Fourier sine transform is found to be,

$$\frac{1}{K}\frac{d\overline{U}_s}{dt} = \frac{b}{h_2} M_0(\lambda_i b) g_s(p,t) - \frac{a}{h_1} M_0(\lambda_i a) f_s(p,t) - \lambda_i^2 \overline{U}_s - p^2 \overline{U}_s + p\overline{V}(\lambda_i, t)$$

and which is expressed as,

$$\frac{d\overline{U}_s}{dt} + K(\lambda_i^2 + p^2)\overline{U}_s =$$

$$= K\left[\frac{b}{h_2} M_0(\lambda_i b) g_s(z,t) - \frac{a}{h_1} M_0(\lambda_i a) f_s(z,t) + p\overline{V}(\lambda_i, t)\right] \tag{41}$$

Now we have the solution of the above ODE given by

$$\overline{U}_s(\lambda_i, p, t) = e^{-K(\lambda_i^2+p^2)t}\Big[K\int_0^t e^{K(\lambda_i^2+p^2)\tau}\Big[\frac{b}{h_2} M_0(\lambda_i b) g_s(z,\tau) - \frac{a}{h_1} M_0(\lambda_i a) f_s(z,\tau) + p\overline{V}(\lambda_i, \tau)\Big]\, d\tau + C\Big]$$

and from the initial condition, we have $C = \overline{I}_s(\lambda_i, p)$, so that

$$\overline{U}_s(\lambda_i, p, t) = e^{-K(\lambda_i^2+p^2)t}\Big[K\int_0^t e^{K(\lambda_i^2+p^2)\tau}\Big[\frac{b}{h_2} M_0(\lambda_i b) g_s(z,\tau) - \frac{a}{h_1} M_0(\lambda_i a) f_s(z,\tau) + p\overline{V}(\lambda_i, \tau)\Big]\, d\tau + \overline{I}_s(\lambda_i, p)\Big] \tag{42}$$

Finally, taking respective inverse transforms will lead to the solution:

$$U(r,z,t) = \frac{2}{\pi}\sum_{i=1}^{\infty} \frac{1}{\mathcal{M}_\nu(\lambda_i)} \left[\int_0^\infty \overline{U}_s(\lambda_i, p, t)\sin(pt)\, dp\right] M_0(\lambda_i r). \tag{43}$$

Special cases:

(i) We consider the previous partial differential equation with the following conditions

$$\left(U + h_1 \frac{\partial U}{\partial r}\right)_{r=a} = 0, \qquad z > 0,\ t > 0$$

$$\left(U + h_2 \frac{\partial U}{\partial r}\right)_{r=b} = \frac{1}{z}, \qquad z > 0, t > 0$$

$$U(r,0,t) = U_0, \quad \text{(const.)} \qquad a < r < b, t > 0$$

$$U(r,z,0) = 0 \qquad a < r < b, z > 0$$

$$U(r,z,t) \to 0 \quad \text{as} \quad z \to \infty$$

then (42) will become

$$\bar{U}_s(\lambda_i,p,t) = e^{-K(\lambda_i^2+p^2)t}\Big[K\int_0^t e^{-K(\lambda_i^2+p^2)x}\Big\{\frac{b}{h_2}M_0(\lambda_i b)\frac{\pi}{2}$$

$$+p\,\frac{U_0}{\lambda_i^2}\Big[\frac{b}{h_2}M_0(\lambda_i b) - \frac{a}{h_1}\,M_0(\lambda_i a)\Big]\Big\}dx\Big]$$

$$= \frac{1-e^{-K(\lambda_i^2+p^2)t}}{\lambda_i^2+p^2}\Big\{\frac{p}{h_2}\Big[\frac{\pi}{2}+\frac{PU_0}{\lambda_i^2}\Big]M_0(\lambda_i b) - \frac{aPU_0}{\lambda_i^2 h_1}\,M_0(\lambda_i a)\Big\}$$

and according to (43)

$$U(r,z,t)=\sum_{i=1}^{\infty}\Big\{\frac{2}{\pi}\int_0^{\infty}\frac{1-e^{-K(\lambda_i^2+p^2)t}}{\lambda_i^2+p^2}\Big\{\frac{p}{h_2}\Big[\frac{\pi}{2}+\frac{PU_0}{\lambda_i^2}\Big]M_0(\lambda_i b)-\frac{aPU_0}{\lambda_i^2 h_1}M_0(\lambda_i a)\Big\}$$

$$\sin(pt)dz\Big\} \times \frac{M_0(\lambda_i r)}{\mathcal{M}_\nu(\lambda_i)}.$$

(ii) For $h_1 \longrightarrow 0$ in (38), that $U\,|_{r=a} = f(z,t)$, our result (43) reduces to one considered in [8, p.43; eq. (36)].

6. Conclusions

Here we have introduced a new finite integral transform (Hankel-type) involving product of Bessel functions as the Kernel. This transform can be used to solve certain class of mixed boundary value problems. As indicated in previous sections 4 and 5, this transform is suitable to solve heat conduction problems in hollow cylinders with radiation (mixed) conditions on both surfaces ($r = a$ and $r = b$). The Hankel-type finite transforms considered earlier in [5, 7, 8, 11] were able to solve problems with both surfaces of the cylinder kept at prescribed temperature or radiation condition on only one surface, $r = b$ [8].

Numerical treatment of the results obtained here can be done by using *"Mathematica"* and [S. L. Kalla and S. Conde: Tables of Bessel Functions and Roots of Transcendental Equations, Univ. Zulia, Venezuela, 1978].

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ЗА ЕДНА ИНТЕГРАЛНА ТРАНСФОРМАЦИЈА КОЈА ГИ СОДРЖИ БЕСЕЛОВИТЕ ФУНКЦИИ

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Резиме

Во оваа работа се проучуваат нови интегрални трансформации, кои вклучуваат комбинација од Беселови функции како јадро. Дадена е инверзна формула и некои нејзини својства. Оваа трансформација може да се употреби за решавање на гранични проблеми од мешан тип. Го разгледавме проблемот за топлопроводност во бесконечен цилиндер ($r = a$, $r = b$, $b > a$) со радијационен тип на гранични услови.

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