

FREE GROUPOIDS WITH AXIOMS OF THE FORM
 $x^{m+1}y = xy$ AND/OR $xy^{n+1} = xy$

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Abstract

The main result of the paper is a canonical description of free objects in the variety $\mathcal{U}(M; N)$ of groupoids with the following axioms:

$$\{x^{m+1} \cdot y = xy \mid m \in M\} \cup \{x \cdot y^{n+1} = xy \mid n \in N\},$$

where M and N are sets of positive integers, such that $M \cup N \neq \emptyset$. Applying the obtained description, corresponding characterization of free subgroupoids of a $\mathcal{U}(M; N)$ -free groupoid is given.

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1. Main results

Throughout the paper $\mathbf{F} = (F; \cdot)$ denotes the absolutely free groupoid (i.e. free groupoid in the variety of all groupoids) with a given basis B . Therefore, \mathbf{F} is injective¹ and B is the set of primes² in F . Moreover, each subgroupoid of \mathbf{F} is free and there exist subgroupoids of \mathbf{F} with infinite basis (see [1], I.1)

There exist $\frac{(2k-2)!}{(k-1)!k!}$ k -th groupoid powers³ $x \mapsto x^k$. In this paper x^k is defined by

$$x^1 = x, \quad x^{k+1} = x^k x,$$

and this is the meaning of the groupoid power in the axioms of $\mathcal{U}(M; N)$.

If $\xi, \eta : F \rightarrow F$ are two transformations on F , then we denote by $\mathbf{F}(\xi, \eta)$ the groupoid (F, \bullet) defined by $x \bullet y = \xi(x)\eta(y)$. We say that the pair ξ, η of transformations on F is *compatible* with \mathbf{F} iff the following two conditions are satisfied:

- 1) $(\forall b \in B) \xi(b) = b = \eta(b)$
- 2) The least subset R of F with the following property:

$$(1) \quad B \subseteq R \ \& \ (\forall t, u \in R)(\xi(t) = t, \eta(u) = u \Rightarrow tu \in R)$$

is a subgroupoid of $\mathbf{F}(\xi, \eta)$.

Here we introduce several notations.

The varieties $\mathcal{U}(M; \emptyset)$, $\mathcal{U}(\emptyset; N)$, $\mathcal{U}(M; N)$, where $M \neq \emptyset$ and $N \neq \emptyset$, are said to be *left*, *right* and *two-sided*, respectively. The variety $\mathcal{U}(M; \emptyset)$ will be also denoted by $\mathcal{U}_l(M)$, and $\mathcal{U}(\emptyset; N)$ by $\mathcal{U}_r(N)$. Further,

$$\mathcal{U}(m_1, m_2, \dots; n_1, n_2, \dots)$$

will be an abbreviation for $\mathcal{U}(\{m_1, m_2, \dots\}; \{n_1, n_2, \dots\})$

We state below the main results of the paper.

Theorem 1. *If B is a nonempty set and M, N are sets of positive integers such that $M \cup N \neq \emptyset$, then there exists a pair (ξ, η) of transformations on F compatible with \mathbf{F} with the following properties:*

¹A groupoid \mathbf{G} is *injective* iff $(\forall x, y, u, v \in G)(xy = uv \Rightarrow x = u \ \& \ y = v)$

²an element $a \in G$ is *prime* in \mathbf{G} iff $a \in G \setminus GG$.

³see [3], III.2, Ex.2, p.125 or [8], pp.39-40

- (i) The restrictions of ξ and η on R are retractions of R .
- (ii) The corresponding groupoid \mathbf{R} is a $\mathcal{U}(M; N)$ -free groupoid with a unique basis B , B being the set of primes in \mathbf{R} .

We say that \mathbf{R} is the $\mathcal{U}(M; N)$ -canonical groupoid with the basis B .

Theorem 2. The class of free objects in a variety $\mathcal{U}(M; N)$ is hereditary iff

$$(M \neq \emptyset, N = \emptyset) \text{ or } (M = \emptyset, 1 \in N).$$

Theorem 3. Let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid with the basis B . If B contains at least two distinct elements or $\mathcal{U}(M; N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m; 1) : m \geq 1\}$, then there exists $\mathcal{U}(M; N)$ -free subgroupoid of \mathbf{H} with infinite basis.

In Section 2 we state some preliminary results, and in Section $i + 2$ we give the proof of Theorem i . Moreover, in Section 4 we describe the family of free subgroupoids of a $\mathcal{U}(M; N)$ -free groupoid in the case when the class of $\mathcal{U}(M; N)$ -free groupoids is not hereditary.

2. Preliminaries

Here we state some properties of the groupoid \mathbf{F} and one of the main results of [6]. Let $x \mapsto |x|$ be the homomorphism of \mathbf{F} into the additive groupoid of positive integers which extends the mapping $B \rightarrow \{1\}$. In other words, we have:

$$(2) \quad (\forall b \in B) |b| = 1,$$

$$(\forall x, y \in F) |xy| = |x| + |y|.$$

(We say that $|t|$ is the *length* of t in \mathbf{F} .)

Below we assume that m is a given positive integer, p, q arbitrary non-negative integers, and i, j, k, \dots arbitrary positive integers. We define two kinds of groupoid powers $x \mapsto x^{(p)}$, $x \mapsto x^{(p)}$ as follows:

$$(3) \quad x^{(0)} = x^{(0)} = x, \quad x^{(p+1)} = (x^{(p)})^{m+1}; \quad x^{(p+1)} = x \underline{x^{(p)} m},$$

where the right-hand side of the last equation has the following meaning:

$$(4) \quad x \underline{y0} = x, \quad x \underline{yp + 1} = (x \underline{yp})y.$$

By induction on the length of elements of F we obtain that, for any $t, u \in F$, $p, q \geq 0$, $i, j \geq 1$, the following relations hold:

$$(5) \quad |t^i| = i|t|; \quad |t^{(p)}| = (m+1)^p|t|; \quad |t^{(p)}| = |t| \sum_{q=0}^p m^q;$$

$$(6) \quad t^{i+1} = u^{j+1} \Rightarrow t = u, \quad i = j;$$

$$(7) \quad t^{(p)} = u^{(p+q)} \iff t = u^{(q)}; \\ (t^{(p)})^{(q)} = t^{(p+q)};$$

$$(8) \quad 1 \leq i < m \Rightarrow (t^{i+1} \neq u^{(p+1)} \ \& \ t \underline{t^{(p)}i} \neq u^{(q+1)});$$

$$(9) \quad t^{(p+1)} = u^{(q+1)} \iff t = u, \quad p = q.$$

One of the main results in [6] is the following

Theorem 2.1. *If M and N are nonempty sets of positive integers, then:*

- (i) $\mathcal{U}(M; \emptyset) = \mathcal{U}(\gcd(M); \emptyset)$;⁴
- (ii) $\mathcal{U}(\emptyset; N) = \mathcal{U}(\emptyset; \langle N \rangle)$;⁵
- (iii) $\mathcal{U}(M; N) = \mathcal{U}(\gcd(M); \gcd(M \cup N))$.

Considering Theorem 2.1 we shall examine three types of $\mathcal{U}(M; N)$ varieties with corresponding canonical sets of axioms, i.e. $\mathcal{U}(\emptyset; S)$, $\mathcal{U}(m; \emptyset)$ and $\mathcal{U}(m; n)$ which will be denoted as $\mathcal{U}_r(S)$, $\mathcal{U}_l(m)$ and $\mathcal{U}(m; n)$, respectively. Here S is the additive groupoid of positive integers generated by N , $m = \gcd(M)$ and $n = \gcd(M \cup N)$ in the case when both $M \neq \emptyset$ and $N \neq \emptyset$.

⁴ $\gcd(M)$ denotes the greatest common divisor of M .

⁵ $\langle N \rangle$ is the subgroupoid of the additive groupoid of positive integers generated by N .

We shall also use the following relations:⁶

$$(10) \quad \mathcal{U}_l(m) \models x^{(p)}y = xy;$$

$$(11) \quad \mathcal{U}_l(m) \models x^{(p)} = x^{(p)};$$

$$(12) \quad \mathcal{U}_l(m) \models x^{pm+i+1} = x^{i+1};$$

$$(13) \quad \mathcal{U}_r(i) \models (x^{i+1})^{j+1} = x^{i+j+1};$$

$$(14) \quad \mathcal{U}(kn; n) \models (x^{in+1})^{kn+1} = x^{in+1}.$$

3. $\mathcal{U}(M; N)$ -canonical groupoids

We assume below that m is a positive integer, and S is an additive groupoid of positive integers.

Define two transformations $\alpha, \beta : F \rightarrow F$, as follows:

$$(15) \quad \alpha(u) = \begin{cases} t, & \text{if } u = t^{(p+1)}, p \geq 0 \\ u, & \text{otherwise} \end{cases}$$

$$(16) \quad \beta(u) = \begin{cases} t, & \text{if } u = t^{i+1}, i \in S \\ u, & \text{otherwise} \end{cases}$$

By (9) and (6), α and β are well defined.

Assume now that M and N are sets of positive integers such that $M \cup N \neq \emptyset$. Using α and β , we define two transformations $\xi, \eta : F \rightarrow F$ for each of the following cases $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}$:

\mathcal{U}_l : If $M \neq \emptyset, N = \emptyset, m = \gcd(M)$, then $\xi = \alpha$ and $\eta = 1_F$;

\mathcal{U}_r : If $M = \emptyset, N \neq \emptyset, S = \langle N \rangle$, then $\xi = 1_F$ and $\eta = \beta$;

\mathcal{U} : If $M \neq \emptyset, N \neq \emptyset, m = \gcd(M), n = \gcd(M \cup N), S = \{in : i \geq 1\}$, then $\xi = \alpha$ and $\eta = \beta$.

⁶ $\mathcal{V} \models \tau_1 = \tau_2$ means: the equation $\tau_1 = \tau_2$ is true in the variety \mathcal{V} .

Clearly, in each of the cases: $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}$ the condition 1) of Section 1 (for the pair (ξ, η) to be compatible with \mathbf{F}) is satisfied. Moreover, according to the condition 2), the corresponding subset R of F is defined as follows: $B \subseteq R$, and

$$(17) \quad \begin{aligned} \mathcal{U}_l &: (\forall v, w \in F)(vw \in R \iff v, w \in R \ \& \ \alpha(v) = v) \\ \mathcal{U}_r &: (\forall v, w \in F)(vw \in R \iff v, w \in R \ \& \ \beta(w) = w) \\ \mathcal{U} &: (\forall v, w \in F)(vw \in R \iff v, w \in R \ \& \ \alpha(v) = v \ \& \ \beta(w) = w). \end{aligned}$$

From (15), (16) and (17), we obtain the following relations:

$$(18) \quad \begin{aligned} \mathcal{U}_l &: v = u^{(p+1)} \Rightarrow (v \in R \iff u \in R \ \& \ \alpha(u) = u); \\ \mathcal{U}_r &: v = u^{i+1} \Rightarrow (v \in R \iff u \in R \ \& \ \beta(u) = u); \\ \mathcal{U} &: v = u^{(p+1)} \Rightarrow (v \in R \iff u \in R, p = 0 \ \& \ \alpha(u) = u); \\ & \quad i \geq 1, v = u^{in+1} \Rightarrow (v \in R \iff u \in R, i \leq k \ \& \ \beta(u) = u), \\ & \quad \text{where } kn = m. \end{aligned}$$

From (18), we obtain:

Proposition 3.1. *The restrictions of ξ and η on R are retractions of R .*

From the definition of the groupoid $\mathbf{F}(\xi, \eta)$ and Proposition 3.1 it follows:

Proposition 3.2. *$\mathbf{R} = (R, \bullet)$ is a subgroupoid of $\mathbf{F}(\xi, \eta)$, and B is the least generating subset of R .*

From (18), the definitions of the pair (ξ, η) and Proposition 3.1 it follows that for each $u \in R$, there exists a unique $t \in R$ and a unique: $p \geq 0$, in the case \mathcal{U}_l ; $i \in S \cup \{0\}$, in the case \mathcal{U}_r ; $s : 0 \leq s \leq k$, in the case \mathcal{U} , such that:

$$(19) \quad \begin{aligned} \mathcal{U}_l &: u = t^{(p)}, \ \alpha(t) = t; \ \mathcal{U}_r : u = t^{i+1}, \ \beta(t) = t; \\ \mathcal{U} &: u = t^{sn+1}, \ \beta(t) = t. \end{aligned}$$

If $v, w \in R$, then $v \bullet w$ can be expressed more explicitly as follows:

$$(20) \quad \begin{aligned} \mathcal{U}_l &: v \bullet w = tw, \ \text{where } v = t^{(p)}, p \geq 0, \alpha(t) = t; \\ \mathcal{U}_r &: v \bullet w = vu, \ \text{where } w = u^{i+1}, i \in S \cup \{0\}, \beta(u) = u; \\ \mathcal{U} &: v \bullet w = \begin{cases} vu, & \text{if } \alpha(v) = v, w = u^{in+1}, 1 \leq i \leq k \\ tw, & \text{if } v = t^{m+1}, \beta(w) = w \\ tu, & \text{if } v = t^{m+1}, w = u^{in+1}, 1 \leq i \leq k \\ vw, & \text{if } \alpha(v) = v, \beta(w) = w. \end{cases} \end{aligned}$$

Now we shall show the following:

Proposition 3.3. $\mathbf{R} \in \mathcal{U}(M; N)$.

Proof. If $u \in R, j \geq 1$, then we denote by u_{\bullet}^j the j -th power of u in \mathbf{R} , i.e.

$$(21) \quad u_{\bullet}^1 = u, \quad u_{\bullet}^{j+1} = (u_{\bullet}^j) \bullet u.$$

(Note that, if $u \in R$, then $u_{\bullet}^j \in R$, but it can happen that $u^j \in F \setminus R$.)

Assuming (19), by (20) we obtain the equalities (22) in the corresponding cases $\mathcal{U}_l, \mathcal{U}_r$ and \mathcal{U} .

$$(22) \quad \begin{aligned} \mathcal{U}_l : \quad u_{\bullet}^j &= \begin{cases} u, & \text{if } j = 1 \\ t \frac{u^j - 1}{u - 1}, & \text{if } 2 \leq j \leq m \\ t \frac{u^r - 1}{u^r - 1}, & \text{if } j = qm + r, 2 \leq r \leq m + 1 \end{cases} \\ \mathcal{U}_r : \quad u_{\bullet}^j &= t^{i+j}, \text{ if } j \geq 1; \\ \mathcal{U} : \quad u_{\bullet}^j &= \begin{cases} u, & \text{if } j = 1 \\ t^{sn+j}, & \text{if } s \geq 1, 1 \leq j \leq (k-s)n + 1. \\ t^{j-(k-s)n}, & \text{if } s \geq 1, (k-s)n + 2 \leq j \leq kn + 1 \end{cases} \end{aligned}$$

Therefore:

$$(23) \quad \begin{aligned} \mathcal{U}_l : \quad u_{\bullet}^{m+1} &= t^{(p+1)}; \\ \mathcal{U}_r : \quad u_{\bullet}^{j+1} &= u^{i+j+1}, \text{ for each } j \in S; \\ \mathcal{U} : \quad u_{\bullet}^{in+1} &= \begin{cases} t^{(i+s)n+1}, & \text{if } i + s \leq k \\ t^{(i+s-k)n+1}, & \text{if } i + s > k, 1 \leq i \leq k. \end{cases} \end{aligned}$$

If $\eta(u) = t \neq u$ (i.e. $u = t^{sn+1}, s \geq 1$) and $i = k$, then in the last case we obtain

$$u_{\bullet}^{m+1} = u.$$

From (20) and (23) we obtain that, for any $v, w \in R$, the following equations hold:

$$\begin{aligned} \mathcal{U}_l : (v_{\bullet}^{m+1}) \bullet w &= v \bullet w; \\ \mathcal{U}_r : v \bullet (w_{\bullet}^{i+1}) &= v \bullet w, \text{ for each } i \in S; \end{aligned}$$

$$\mathcal{U} : (v \bullet^{m+1}) \bullet w = v \bullet w = v \bullet (w \bullet^{n+1}).$$

Therefore, we have $\mathbf{R} \in \mathcal{U}_l(m)$, $\mathbf{R} \in \mathcal{U}_r(S)$, $\mathbf{R} \in \mathcal{U}(kn; n)$ in the cases: $\mathcal{U}(M; N) = \mathcal{U}_l(m)$, $\mathcal{U}(M; N) = \mathcal{U}_r(S)$, $\mathcal{U}(M; N) = \mathcal{U}(kn; n)$, respectively. \square

The following statement will complete the proof of Theorem 1.

Proposition 3.4. *Let $\mathbf{G} = (G; \cdot) \in \mathcal{U}(M; N)$. If $\lambda : B \rightarrow G$ is a mapping, and $\varphi : \mathbf{F} \rightarrow \mathbf{G}$ the homomorphism which extends λ , then the restriction of φ on R is a homomorphism from \mathbf{R} into \mathbf{G} .*

Proof. It suffices to show the equality $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, for each $v, w \in R$ such that $v \bullet w \neq vw$.

Then, in the case $\mathcal{U}(M; N) = \mathcal{U}_l(m)$, we have $v = t^{(p+1)}$ for a unique pair (t, p) , where $t \in R, p \geq 0, \alpha(t) = t$, and $v \bullet w = tw$. Therefore, we have:

$$\varphi(v \bullet w) = \varphi(tw) = \varphi(t)\varphi(w) = \varphi(t)^{(p+1)}\varphi(w).$$

Then, by (11) we have:

$$\varphi(t)^{(p+1)}\varphi(w) = \varphi(t)^{(p+1)}\varphi(w) = \varphi(t^{(p+1)})\varphi(w) = \varphi(v)\varphi(w).$$

In the case $\mathcal{U}(M; N) = \mathcal{U}_r(S)$ we have: $w = t^{i+1}$, for a unique pair (t, i) , where $t \in R, \beta(t) = t, i \in S$. Therefore,

$$\varphi(v \bullet w) = \varphi(vt) = \varphi(v)\varphi(t) = \varphi(v)\varphi(t)^{i+1} = \varphi(v)\varphi(t^{i+1}) = \varphi(v)\varphi(w).$$

In a similar way, we obtain that $\varphi(v \bullet w) = \varphi(v)\varphi(w)$, in the case \mathcal{U} . \square

By Propositions 3.2–3.4, \mathbf{R} is a $\mathcal{U}(M; N)$ -free groupoid with the unique basis B , i.e. we have completed the proof of Theorem 1.

We say that the formula: $x^{m+1} \cdot y = xy$ ($x \cdot y^{n+1} = xy$) is a *left* (a *right*) equation; a left or a right equation is called *equation*. It is well known that an equation holds in a variety $\mathcal{U}(M; N)$ iff it is satisfied in each $\mathcal{U}(M; N)$ -free groupoid. Therefore, the following statement describes the set of equations in a variety $\mathcal{U}(M; N)$.

Proposition 3.5. *Let \mathbf{H} be a free groupoid in the variety $\mathcal{U}(M; N)$. Then the following statements hold.*

- (i) If $M \neq \emptyset$, $N = \emptyset$, $\gcd(M) = m$, then a left equation $x^{m+1}y = xy$ holds in \mathbf{H} iff $m|n$; no right equation holds in \mathbf{H} .
- (ii) If $M = \emptyset$, $N \neq \emptyset$, then the right equation $xy^{j+1} = xy$ holds in \mathbf{H} iff $j \in \langle N \rangle$; no left equation holds in \mathbf{H} .
- (iii) If $M \neq \emptyset$, $N \neq \emptyset$, $\gcd(M) = m$, $n = \gcd(M \cup N)$, then $x^{i+1}y = xy$ iff $m|i$ and $xy^{j+1} = xy$ iff $n|j$ hold in \mathbf{H} .

Proof. Let \mathbf{R} be a $\mathcal{U}(M; N)$ -canonical groupoid with the basis B , and $a, b \in B$.

Then:

- (i) If $M \neq \emptyset$, $N = \emptyset$, $\gcd(M) = m$, then
 $(a^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$;
 $a \bullet b^{j+1} \neq ab = a \bullet b$ for each $j \geq 1$.
- (ii) If $M = \emptyset$, $N \neq \emptyset$ and $S = \langle N \rangle$, then:
 $a^{i+1} \bullet b = a^{i+1}b \neq ab = a \bullet b$;
and, if $j \geq 1$, then $a \bullet (a^{j+1}) = ab = a \bullet b$ iff $j \in S$.
- (iii) If $M \neq \emptyset$, $N \neq \emptyset$, $\gcd(M) = m$, $n = \gcd(M \cup N)$, $i, j \geq 1$, then:
 $(a^{i+1}) \bullet b = ab = a \bullet b$ iff $m|i$,
 $a \bullet (b^{j+1}) = ab = a \bullet b$ iff $n|j$. \square

Having in mind the definitions of the transformations ξ, η in each of the cases $\mathcal{U}_l(m), \mathcal{U}_r(S)$ and $\mathcal{U}(kn; n)$, as a corollary of Theorem 1 the following statement can also be obtained.

Proposition 3.6. *If \mathbf{H} is a $\mathcal{U}(M; N)$ -free groupoid with the basis B , then there exist retractions γ and δ of H with the following properties:*

- (i) B is the set of primes in \mathbf{H} , and $B \subseteq \text{im}\gamma \cap \text{im}\delta$;
(If $x \in \text{im}\gamma \cap \text{im}\delta$, then we say that x is a base in \mathbf{H})
- (ii) $(\forall x, y \in H)xy = \gamma(x)\delta(y)$;
 $((\gamma(x), \delta(y)))$ is the pair of divisors of xy in \mathbf{H} ; i.e. $\gamma(x)$ is the left and $\delta(y)$ the right divisor of xy .)

- (iii) There exists a mapping $x \mapsto |x|$ from H into the set of positive integers with the following properties:

$$|xy| = |\gamma(x)| + |\delta(y)|,$$

$$\gamma(x) \neq x \iff |\gamma(x)| < |x|; \delta(x) \neq x \iff |\delta(x)| < |x|,$$

for any $x, y \in H$.

Proof. If \mathbf{R} is the $\mathcal{U}(M; N)$ -canonical groupoid with the basis B , then there exists a unique isomorphism $\varphi: \mathbf{R} \rightarrow \mathbf{H}$ such that $\varphi(b) = b$, for each $b \in B$. Defining $\gamma, \delta: H \rightarrow H$ by: $\gamma(x) = \xi(\varphi^{-1}(x))$, $\delta(x) = \eta(\varphi^{-1}(x))$, we obtain two retractions γ, δ of H such that (i)–(iii) hold, where the length of $x \in H$ is defined by $|x| = |\varphi^{-1}(x)|$. \square

In each of the cases $\mathcal{U}_l(m), \mathcal{U}_r(S), \mathcal{U}(kn; n)$, the results of Proposition 3.6 can be stated more explicitly as follows.

3.6. $\mathcal{U}_l(m)$.

- (i) $\gamma = \alpha, \delta = 1_H$;
- (ii) $y \in H$ is a base in \mathbf{H} iff $y \in \text{im}\gamma$; for each $x \in H$ there exists a unique $y = \text{bs}(x)$ (the base of x) and unique $p = \text{exp}(x) \geq 0$ (the exponent of x) such that $x = y^{(p)}$.
- (iii) $\text{bs}(x)$ is the left (and y the right) divisor of xy .
- (iv) If b is a base in \mathbf{H} , and $1 \leq i < m$, $p \geq 0$, then $c = b \underline{b^{(p)}i}$ is also a base in \mathbf{H} ; $b \underline{b^{(p)}i - 1}$ is the left and $b^{(p)}$ the right divisor of c ; in the same case $b \underline{b^{(p)}m - 1}$ is the left and $b^{(p)}$ the right divisor of $b^{(p+1)}$.
- (v) If $x \in H, 1 \leq i \leq j \leq m + 1$, then $x^i = x^j \Rightarrow i = j$.

3.6. $\mathcal{U}_r(N)$.

- (i) $\gamma = 1_H, \delta = \beta$;
- (ii) y is a base in \mathbf{H} iff $y \in \text{im}\delta$; for each $x \in H$ there exists a unique base y , and a unique $q \in \{0\} \cup \langle N \rangle$, such that $x = y^{q+1}$.

- (iii) The left divisor of xy is x and its right divisor is $\text{bs}(y)$. Thus,
 $xy = uv \iff x = u, \text{bs}(y) = \text{bs}(v)$.

3.6. $\mathcal{U}(kn; n)$.

- (i) x is a base in \mathbf{H} iff $x^{n+1} \neq x$; for each $x \in H$ there exists a unique y (the *base of* x) and a unique $i \in \{0, 1, \dots, k\}$ (the *exponent of* x) such that $x = y^{in+1}$; x is a *left base* in \mathbf{H} iff $i \neq k$, where i is the exponent of x .
- (ii) For any $x \in H$, $\delta(x)$ is the base of x , and

$$\gamma(x) = \begin{cases} x, & \text{if } x \text{ is the left base} \\ y, & \text{if } y \text{ is the base of } x, \text{ and } x = y^{m+1} \end{cases}$$

$$\delta(x) = \begin{cases} y, & \text{if } x = y^{in+1}, 0 \leq i \leq k, \\ x, & \text{otherwise.} \end{cases}$$

- (iii) $\gamma(x)$ is the left and $\delta(y)$ the right divisor of xy .

4. Free subgroupoids of $\mathcal{U}(M; N)$ -free groupoids

We shall describe the set of pairs (M, N) of sets of positive integers such that the variety $\mathcal{U}(M; N)$ is hereditary, i.e. we shall prove Theorem 2.

Proposition 4.1. *For any $m \geq 1$, the class of free objects in the variety $\mathcal{U}_l(m)$ is hereditary.*

Proof. Let \mathbf{Q} be a subgroupoid of a $\mathcal{U}_l(m)$ -free groupoid \mathbf{H} . We have to show that the set P of prime elements of \mathbf{Q} is nonempty, and that \mathbf{Q} is $\mathcal{U}_l(m)$ -free with the basis P . The proof will be given in several steps, where induction on $|x|$, for $x \in Q$, will be used.

- 1) If $a \in Q \setminus P$ and c is the right divisor of a in \mathbf{H} , then $c \in Q$.
- 2) Let $a = b^{(p)} \in Q$, where b is the base of a in \mathbf{H} . If q is the least non-negative integer such that $b' = b^{(q)} \in Q$, then we say that b' is the base of a in \mathbf{Q} . Then, if $q \geq 1, b' \in P$.

By 1) and 2) we obtain:

- 3) $P \neq \emptyset$, and P is the least generating subset of \mathbf{Q} .

4) If $c, d \in Q$, and b' is the base of c in \mathbf{Q} , then we say that (b', d) is the pair of divisors of cd in \mathbf{Q} . Then: $|d| < |cd|$, and: b' is prime or $|cd| = |b'| + |d|$.

5) Assume that $\mathbf{G} \in \mathcal{U}_l(m)$ and $\lambda : P \rightarrow G$ is a given mapping. There is a (unique) homomorphism $\varphi : Q \rightarrow G$ such that $\lambda = \varphi|_P$ is the restriction of φ on P . Namely, if $x \in Q$ is such that $|x| = \min\{|y| : y \in Q\}$, then $y \in P$, and thus $\varphi(x) = \lambda(x)$ is well defined. Assume that for each $x \in Q$, such that $|x| \leq i$, $\varphi(x) \in G$ is well defined and, moreover, if (y, z) is the pair of divisors of x in \mathbf{Q} , then $\varphi(y), \varphi(z)$ are well defined, and $\varphi(x) = \varphi(y)\varphi(z)$.

Let $v \in Q \setminus P$ be such that $|v| = i + 1$, and (t, u) be the pair of divisors of v in \mathbf{Q} . Then $\varphi(t)$ and $\varphi(u)$ are well defined, and thus we can define $\varphi(v)$ by $\varphi(v) = \varphi(t)\varphi(u)$. Then $\varphi : \mathbf{Q} \rightarrow \mathbf{G}$ is a homomorphism which extends λ . \square

Proposition 4.2. *The class of free objects in the variety $\mathcal{U}_r(1)$ is hereditary.*

Proof. This statement is one of the main results of [4], and it is also a corollary of Proposition 4.1. Namely, let $\mathbf{G} = (G, \cdot)$ be a given groupoid, and the groupoid $\mathbf{G}^{\text{op}} = (G, \circ)$ be defined by $x \circ y = yx$. Then, $\mathbf{G} \in \mathcal{U}_r(1) \iff \mathbf{G}^{\text{op}} \in \mathcal{U}_l(1)$, and \mathbf{H} is $\mathcal{U}_r(1)$ -free iff \mathbf{H}^{op} is $\mathcal{U}_l(1)$ -free. \square

Proposition 4.3. *If N is a nonempty set of positive integers and $1 \notin N$, then the class of free objects in the variety $\mathcal{U}_r(N)$ is not hereditary.*

Proof. Let $n = \min(N)$, and let \mathbf{H} be a $\mathcal{U}_r(N)$ -free groupoid with the basis B . Consider the subgroupoid \mathbf{Q} generated by $\{b^n, b^{n+1}\}$, where $b \in B$. Then b^n is the unique prime in \mathbf{Q} , and $\{b^n\}$ does not generate \mathbf{Q} , which implies that \mathbf{Q} is not free⁷. \square

Proposition 4.4. *If $M \neq \emptyset, N \neq \emptyset$, then the class of free objects in the variety $\mathcal{U}(M; N)$ is not hereditary.*

Proof. Let $m = \gcd(M), n = \gcd(M \cup N)$ and let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid with the basis B . If $b \in B$, and \mathbf{Q} is the subgroupoid generated by $\{b^{n+1}\}$, then the set of primes in \mathbf{Q} is empty. (Namely, $(b^{n+1})^{m+1} = b^{n+1}$, which implies that b^{n+1} is not a prime in \mathbf{Q} .) \square

Theorem 2 is a corollary of Propositions 4.1–4.4.

⁷Here, and further on in Section 4, if \mathbf{H} is $\mathcal{U}(M; N)$ -free groupoid, and \mathbf{Q} is a subgroupoid of \mathbf{H} , we will write " \mathbf{Q} is free" instead of " \mathbf{Q} is $\mathcal{U}(M; N)$ -free".

Proposition 4.5. *Let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid and \mathbf{Q} a subgroupoid of \mathbf{H} , such that:*

$$(24) \quad (\forall x \in H)(x \in Q \Rightarrow \text{bs}(x) \in Q).$$

Then \mathbf{Q} is free.

Proof. From (24) it follows that if $a \in QQ$ and (c, d) is the pair of divisors of a in \mathbf{H} , then $c, d \in Q$, and moreover the following equation holds:

$$|a| = |c| + |d|.$$

This implies that the set P of primes in \mathbf{Q} is nonempty and generates \mathbf{Q} . In the same way as 5) in the proof of Proposition 4.1, one can show that \mathbf{Q} is free with the basis P . \square

In the next three statements we describe free subgroupoids of $\mathcal{U}(M; N)$ -free groupoids when the class of $\mathcal{U}(M; N)$ -free groupoids is not hereditary.

Proposition 4.6. *Let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid, where $M \neq \emptyset, N \neq \emptyset$, and \mathbf{Q} be a subgroupoid of \mathbf{H} . If \mathbf{Q} does not satisfy (24), then \mathbf{Q} is not free.*

Proof. Let $m = \text{gcd}(M), n = \text{gcd}(M \cup N), m = kn$, and let $a \in Q$ be such that $b \notin Q$, where b is the base of a in \mathbf{H} . Then, there exists an $i \in \{1, 2, \dots, k\}$, such that $a = b^{in+1}$. Then:

$$\begin{aligned} a^2 &= b^{in+2}, a^3 = b^{in+3}, \dots, a^{(k-i)n+1} = b^{kn+1} = b^{m+1}, \\ a^{(k-i)+2} &= b^2, a^{(k-i)n+3} = b^3, \dots, a^{(k-i)n+n} = b^n, \\ a^{(k-i+1)n+1} &= b^{n+1}, \dots, a^{(k-i+2)n+1} = b^{2n+1}, \dots, a^{kn} = b^{in}, a^{kn+1} = a \end{aligned}$$

are elements of Q . Thus, $b^2, b^3, \dots, b^{m+1} \in Q$, but $b \notin Q$. From the equality $(b^{m+1})^{m+1} = b^{m+1}$ it follows that b^{m+1} is not a base, and if \mathbf{Q} were free, a base c in \mathbf{Q} and $j \in \{1, 2, \dots, k\}$ would exist, such that $b^{m+1} = c^{in+1}$, which would imply $i = k, c = b \in Q$, i.e. we would obtain a contradiction. \square

Proposition 4.7. *Let \mathbf{Q} be a subgroupoid of a $\mathcal{U}_r(N)$ -free groupoid, where $n = \min(N) \in N$, and let, for $x \in Q$, $j(x)$ be defined as follows:*

$$(25) \quad j(x) = \min\{s : (\text{bs}(x))^{s+1} \in Q, s \geq 0\}.$$

Then, \mathbf{Q} is free iff

$$(26) \quad (\forall x \in Q) n|j(x).$$

Proof. Assuming the condition (26), in the same way as in the proof of Proposition 4.1, one can show that \mathbf{Q} is free.

Thus, we can assume that there exists an $a \in Q$, such that n is not a divisor of $j = j(a)$. Then, b^{j+1} is a prime in \mathbf{Q} , where $b = \text{bs}(a)$ is the base of a ; moreover, then $b^{j+\nu} \in Q$, for any $\nu \geq 1$. Denote by \mathbf{T} the subgroupoid of \mathbf{Q} generated by the set P of primes in \mathbf{Q} . Thus: $T = \bigcup\{P_\nu : \nu \geq 1\}$, where $P_1 = P$, $P_{\nu+1} = P_\nu \cup \{xy : x, y \in P_\nu\}$. Then, if $i \geq 1$ is such that $in > j$, $b^{in+1} \in Q \setminus T$, and therefore the set P of primes in \mathbf{Q} does not generate \mathbf{Q} . \square

Proposition 4.8. *Let \mathbf{H} be a $\mathcal{U}_r(N)$ -free groupoid, where $\text{gcd}(N) \notin N$. A subgroupoid \mathbf{Q} of \mathbf{H} is free iff \mathbf{Q} satisfies the condition (24).*

Proof. Denote by S the additive groupoid of positive integers generated by N . Then $n = \text{gcd}(S) = \text{gcd}(N) \notin S$. If \mathbf{Q} satisfies the condition (24), then, by Proposition 4.5, \mathbf{Q} is $\mathcal{U}_r(N)$ -free.

Assume that there exists an $a \in Q$, such that $b \notin Q$, where b is the base of a in \mathbf{H} . Therefore, there exists an $s \in S$, such that $a = b^{s+1}$. Let

$$(27) \quad i = \min\{\nu : b^{\nu+1} \in Q, \nu \geq 1\}.$$

Then $i \geq 1$, and $b^{i+1} \in Q$; moreover b^{i+1} is prime in \mathbf{Q} . If $i \notin S$, then one can show that \mathbf{Q} is not $\mathcal{U}_r(N)$ -free in the same way as in Proposition 4.7, in the case "n is not a divisor of j".

Thus we can assume that $i \in S$. Then $b^{i+\nu} \in Q$, for any $\nu \geq 1$.

Let s be the least element of S such that $s + \nu n \in S$, for any $\nu \geq 1$. (See [6, Lemma 1.6.iii] or [7].) Then $j = s + (i - n) \in S$, but $j - i = s - n \notin S$. Let k be the least element of S such that $i < k \leq j$, and $j - k \in S \cup \{0\}$. Then: $i + j = k + i + \alpha$, where $\alpha \in S \cup \{0\}$.

Assume that \mathbf{Q} is free. Then b^{i+1} and b^{k+1} are different bases in \mathbf{Q} , but

$$(b^{i+1})^{j+1} = b^{i+j+1} = b^{i+k+\alpha+1} = (b^{k+1})^{i+\alpha+1},$$

which is impossible. \square

5. Ranks of free subgroupoids of $\mathcal{U}(M; N)$ -free groupoids

We shall first consider $\mathcal{U}(M; N)$ -free groupoids with one element basis $B = \{b\}$ in the cases $\mathcal{U}_l(1), \mathcal{U}_r(1)$ and $\mathcal{U}(1; 1)$ and then prove Theorem 3.

Proposition 5.1. *If Z^+ is the set of positive integers, then the groupoid (Z^+, \bullet) defined by $i \bullet j = i + 1$ is $\mathcal{U}_l(1)$ -free groupoid with the basis $\{1\}$. If \mathbf{Q} is a subgroupoid of (Z^+, \bullet) and m is the least element of \mathbf{Q} , then \mathbf{Q} is a $\mathcal{U}_l(1)$ -free groupoid with the basis $\{m\}$. The groupoid $(Z^+, \bullet)^{op}$ is $\mathcal{U}_r(1)$ -free with the basis $\{1\}$.*

Proposition 5.2. *The groupoid $\mathbf{H} = (\{1, 2, \dots, m, m + 1\}, \bullet)$, defined by*

$$i \bullet j = \begin{cases} i + 1, & \text{for } i \leq m \\ 2, & \text{for } i = m + 1 \end{cases}$$

is $\mathcal{U}(1; 1)$ -free groupoid with the basis $\{1\}$. If \mathbf{Q} is a proper subgroupoid of \mathbf{H} , then \mathbf{Q} is not $\mathcal{U}(1; 1)$ -free.

Proof. $Q = \{2, 3, \dots, m, m + 1\}$ is the unique proper subgroupoid in \mathbf{H} . The set of primes in \mathbf{Q} is empty, and thus, \mathbf{Q} is not $\mathcal{U}(1; 1)$ -free. \square

Proposition 5.3. *Let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid with the basis $\{b\}$, where*

$$(28) \quad \mathcal{U}(M; N) \notin \{\mathcal{U}_l(1), \mathcal{U}_r(1)\} \cup \{\mathcal{U}(m; 1) : m \geq 1\},$$

and let $A = \{a_i : i \geq 1\} \subseteq H$ be defined as follows:

$$(29) \quad a_1 = b^2, \quad a_{i+1} = ba_i.$$

If \mathbf{Q} is a subgroupoid of \mathbf{H} generated by A , then \mathbf{Q} is $\mathcal{U}(M; N)$ -free with the basis A , and $a_i = a_j \Rightarrow i = j$.

Proof. Assuming that $\mathbf{H} = \mathbf{R}$ is the $\mathcal{U}(M; N)$ -canonical groupoid with the basis $\{b\}$, we obtain that $a_i = a_j \Rightarrow i = j$, i.e. A is an infinite subset of H . Moreover, for each $i \geq 1$, b is the left divisor for a_i , and $b \notin Q$. This implies that a_i is prime in \mathbf{Q} .

It remains to show that \mathbf{Q} is $\mathcal{U}(M; N)$ -free with the basis A .

If $\mathcal{U}(M; N) = \mathcal{U}_l(m)$, by Proposition 4.1 we obtain that \mathbf{Q} is $\mathcal{U}(M; N)$ -free with the basis A .

Assume now that $\mathcal{U}(M; N) = \mathcal{U}_r(S)$, $1 \notin S = \langle N \rangle$, and that $d = c^{i+1} \in Q$, where $i \in S$, and c is a base in \mathbf{H} . Then $d \notin A$, and therefore $c^i \in Q$. Continuing in such a way, we would obtain $c \in Q$. Thus by Proposition 4.5, (24) is satisfied, and thus \mathbf{Q} is $\mathcal{U}_r(S)$ -free with the basis A .

Finally, let $\mathcal{U}(M; N) = \mathcal{U}(kn; n), n \geq 2$. The fact that A generates \mathbf{Q} implies:

$$Q = \bigcup \{A_i : i \geq 1\}, \text{ where } A_1 = A, A_{i+1} = A_i \cup \{xy : x, y \in A_i\}.$$

Assume that $d = c^{in+1} \in Q$, where $1 \leq i \leq k$, and c is a base in \mathbf{H} . Let s be the least positive integer, such that $d \in A_{s+1} \setminus A_s$. Such an s exists as $d \notin A$. Thus, there exist $d', d'' \in A_s$, such that $c^{in+1} = c^{in}c = d = d'd''$, and therefore $c^{in} = d', d'' = c^{jn+1}$ for some $0 \leq j \leq k$. So, $c^{in} \in Q \setminus A$; then, by the same argument, $c^{in-1} \in Q$, etc., and by an obvious induction we obtain that $c \in Q$. Therefore, \mathbf{Q} is free. \square

We note that in the case $\mathcal{U}(M; N) = \mathcal{U}_l(m), \mathbf{Q}$ satisfies the relation (24). Also:

- $\mathcal{U}(M; N) = \mathcal{U}_l(1) \Rightarrow A = \{b^2\}$, and \mathbf{Q} is $\mathcal{U}_l(1)$ -free with the basis A ;
- $\mathcal{U}(M; N) = \mathcal{U}_r(1) \Rightarrow Q = A$, and \mathbf{Q} is $\mathcal{U}_r(1)$ -free with the basis $\{b^2\}$;
- $\mathcal{U}(M; N) = \mathcal{U}(m; 1) \Rightarrow A = \{b^2\}, Q = \{b^2, b^3, \dots, b^{m+1}\}$, and \mathbf{Q} is not $\mathcal{U}(m; 1)$ -free.

The proof of the following statement is the same as the proof of Proposition 5.3, and moreover, the assumption (28) is not necessary.

Proposition 5.4. *Let \mathbf{H} be a $\mathcal{U}(M; N)$ -free groupoid with the basis $\{a, b\}$, $a \neq b$, and let $C = \{c_i : i \geq 1\}$ be defined as follows:*

$$(30) \quad c_1 = ab, \quad c_{i+1} = ac_i.$$

Then $c_i = c_j \Rightarrow i = j$, and the subgroupoid \mathbf{Q} of \mathbf{H} generated by C is $\mathcal{U}(M; N)$ -free with the basis C .

This completes the proof of Theorem 3.

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