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FREE GROUPOIDS WITH $x^n = x$ II

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Abstract. We study free objects in a set of varieties of groupoids with an axiom of the form $x^n = x$, where x^n is an arbitrary n-th power of x. Powers are considered as elements of the absolutely free groupoid **E** with a one-element basis. The description of free objects for reduced elements in **E** is given.

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1. Main results

This paper is organized as follows. In Section 1 we give our main results in four theorems. Their proofs are presented in Section 3 using several necessary propositions as well as properties of the groupoids **E** and **F** given in Section 2.

We denote by $\mathbf{F} = (F, \cdot)$ an absolutely free groupoid with a basis B, that is, a groupoid free in the variety of groupoids.

Recall that the conjunction of the following two statements characterizes **F** :

$$(\forall x, y, u, v \in F)\ (xy = uv \Rightarrow x = u \text{ and } y = v),$$
$$(1) \qquad (\forall b \in B)(\forall x, y \in F) b \neq xy,$$

(see for example [1, I.1]). Also, we denote by $\mathbf{E} = (E, \cdot)$ an absolutely free groupoid with a one-element basis $\{e\}$. Elements of F will be denoted by $t, u, v, w, \ldots$ and elements of E by $f, g, h, \ldots$. For every $v \in F$ we define a finite subset $P(v)$ as follows:

$$b \in B \Rightarrow P(b) = \{b\};\ P(tu) = \{tu\} \cup P(t) \cup P(u).$$

If $u \in P(v)$, then we say that u is a *part* of v, and if, in addition, $u \neq v$, then u is a *proper part* of v. This makes meaningful the notion "part of an $f \in E$", as well.

Let $\mathbf{G} = (G, \cdot)$ be a groupoid, $a \in G$ and $f \in E$. If φ_a is the homomorphism from **E** into **G** such that $\varphi_a(e) = a$, then we will write $f^{\mathbf{G}}(a)$ instead of $\varphi_a(f)$. Thus,

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$$(fg)^{\mathbf{G}}(x) = f^{\mathbf{G}}(x)g^{\mathbf{G}}(x),$$

for any $f, g \in E$, $x \in G$. Following this notation we will write $f(u)$ instead of $f^{\mathbf{F}}(u)$ when $\mathbf{G} = \mathbf{F}$ and $f(g)$ instead of $f^{\mathbf{E}}(g)$ in the case $\mathbf{G} = \mathbf{E}$. Thus,

$$f(e) = e(f) = f,$$

for any $f \in E$.

We say that f is *reduced* in $\mathbf{E}$ if $f \neq e$ and

(2) $$f = g(h) \Rightarrow g = e \quad \text{or} \quad h = e.$$

In what follows we denote by $\mathcal{V}^f$ the variety of all groupoids $\mathbf{G} = (G, \cdot)$ such that

$$f^{\mathbf{G}}(x) = x$$

for each $x \in G$. We usually assume that $f \neq e$ since $\mathcal{V}^e$ is the variety of all the groupoids.

If k is a positive integer then e^k is defined by

(3) $$e^1 = e, \quad e^{k+1} = e^k e.$$

Also, in the same sense,

$$u^1 = u, \quad u^{k+1} = u^k u$$

for any $u \in F$.

Now, we will state the main results of the paper.

Theorem 1. *Let $f \in E$, $f \neq e$, and let R_f be the set of all elements $u \in F$ such that, for any $t \in F$, $f(t)$ is not part of u. For $v, w \in R_f$ define $v \bullet w$ by*

(4) $$v \bullet w = \begin{cases} vw, & vw \in R_f \\ t, & vw = f(t). \end{cases}$$

Then:

(i) $\mathbf{R}_f = (R_f, \bullet)$ *is a groupoid, and B is the least generating set of R_f;*

(ii) *If $\mathbf{G} = (G, \cdot) \in \mathcal{V}^f$ and $\lambda : B \to G$ is an arbitrary mapping, then there is a homomorphism $\psi : \mathbf{R}_f \to \mathbf{G}$ which extends λ.*

Theorem 2. *If f is reduced in $\mathbf{E}$, then $\mathbf{R}_f$ is a free groupoid in $\mathcal{V}^f$ with the unique basis B.*

Theorem 3. *If $f = (e^m)^n$, then $\mathbf{R}_f$ is a free groupoid in $\mathcal{V}^f$ if and only if $m \leq n$ or $n = 1$.*

In order to describe free objects in $\mathcal{V}^f$, where $f = (e^m)^n$ and $2 \leq n < m$, let us define new kinds of "powers" $e^{<p>}$, for any non-negative integer p, as follows:

$$(5) \qquad e^{<0>} = e, \quad e^{<1>} = e^m, \quad e^{<p+2>} = e^{<p>}\underline{e^{<p+1>}m-n}.$$

The underlined part in (5) has the following meaning:

$$(6) \qquad \underline{xy1} = xy, \quad \underline{xyk+1} = (\underline{xyk})y.$$

More generally, if $u \in F$, then

$$u^{<0>} = u, \quad u^{<1>} = u^m, \quad u^{<p+2>} = u^{<p>}\underline{u^{<p+1>}m-n}.$$

Theorem 4. *Let $f = (e^m)^n$, where $2 \leq n < m$, and let $S \subset F$ be defined by*

$$(7) \qquad S = \{u \in F \mid (\forall t \in F,\ p \geq 0)((t^{<p+1>})^n \notin P(u)\}.$$

For any $v, w \in S$ define $v * w$ by

$$(8) \qquad v * w = \begin{cases} vw, & vw \in S \\ t^{<p>}, & vw = (t^{<p+1>})^n. \end{cases}$$

Then $\mathbf{S} = (S, *)$ is a groupoid such that

$$S \subset R_f \ \text{ and } \ (v, w, v \bullet w \in S \Rightarrow v * w = v \bullet w).$$

Moreover, $\mathbf{S}$ is a free groupoid in $\mathcal{V}^f$ with the unique basis B.

Remarks. 1. If $k \geq 2$, then e^k is reduced in $\mathbf{E}$, and this implies that the main result of [2] is a corollary of Theorem 2.

2. If $n \geq 2$, $m \geq 2$, then $f = g(h)$, where $f = (e^m)^n$, $g = e^n$, $h = e^m$, i.e. f is not reduced in $\mathbf{E}$. This fact and Theorem 3 imply that the condition "f is reduced" is not necessary in Theorem 2.

3. If we define the sets

$$\begin{aligned} H &= \{f \in E \mid f \text{ is not reduced, and } R_f \in \mathcal{V}^f\}, \\ L &= \{f \in E \mid f \text{ is not reduced, and } R_f \notin \mathcal{V}^f\}, \end{aligned}$$

then, by Theorem 3, both the sets H and L are infinite. This result suggests the problem of convenient characterization of H in E. Also, if $f \in L$, we can ask for variants of Theorem 4.

2. Some properties of E and F

In this section we state some properties of the groupoids $\mathbf{E}$ and $\mathbf{F}$ which will be used in the next section for the proofs of the main results presented in Section 1.

First, we denote by $x \mapsto |x|$ the homomorphism from $\mathbf{F}$ into the additive groupoid of positive integer, which extends the mapping $B \to \{1\}$. Therefore,

$$|b| = 1, \quad |tu| = |t| + |u| \tag{9}$$

for any $b \in B$, $t, u \in F$.

We say that $|u|$ is the *length* of u. As a special case, we have that $|f|$ is a positive integer for any $f \in E$, and

$$|e| = 1, \quad |fg| = |f| + |g|, \tag{10}$$

for any $f, g \in E$.

By (9), (10) and the corresponding induction on lengths, the following properties can be easily shown:

$$|f(u)| = |f||u|, \quad |f(g)| = |f||g|, \tag{11}$$

$$f(t) = g(u) \text{ and } (|f| = |g| \text{ or } |t| = |u|) \Rightarrow (f = g \text{ and } t = u), \tag{12}$$

for any $t, u \in F$ and $f, g \in E$.

Using the relations from Sections 1 and 2, it is easy to check the following four assertions.

Proposition 2.1. *If $f, g \in E$, $t, u \in F$ are such that $f(t) = g(u)$ and $|t| \leq |u|$, then there exists a unique $h \in E$ such that:*

$$f = g(h), \quad u = h(t).$$

Proposition 2.2. *$(e^m)^n$ is reduced if and only if*

$$m = 1, n \geq 2 \quad or \quad n = 1, m \geq 2.$$

Proposition 2.3. *If $\varphi : \mathbf{G} \to \mathbf{G}'$ is a homomorphism, then*

$$\varphi(f^{\mathbf{G}}(a)) = f^{\mathbf{G}'}(\varphi(a))$$

for any $f \in E$, $a \in G$.

Proposition 2.4. *If p, q are non-negative integers and $t, u \in F$, then $|t^{<p>}| < |t^{<p+1>}|$ and*

$$t^{<p+1>} = u^{<q+1>} \Rightarrow t = u \text{ and } p = q.$$

In the proof of Theorem 4 we will use new kinds of powers $x \mapsto x^{(p)}$, assuming that $m \geq 3$ is a given integer. Namely:

$$x^{(0)} = x, \quad x^{(p+1)} = (x^{(p)})^m. \tag{13}$$

We have the following.

Proposition 2.5. *If $2 \leq n < m$, $f = (e^m)^n$ and $\mathbf{G} = (G, \cdot) \in \mathcal{V}^f$, then:*

$$x^{<p+1>} = x^{(p+1)}, \tag{14}$$

for each $p \geq 0$ and $x \in G$.

The following statement is a corollary of P. Hall's result stated in [3, p. 125] and [4, pp. 39–40].

Proposition 2.6. *If $E_n = \{f \in E \mid |f| = n\}$, then E_n is a finite subset of E and E_n consists of exactly $(2n-2)!/\big(n!(n-1)!\big)$ elements.*

Actually, each element of E_n can be considered as an n-th groupoid power.

3. Proofs of Theorems

In the sequel we write R instead of R_f. First, we give the following two obvious statements:

$$B \subset R \subset F,$$

$$(\forall v, w)\{vw \in R \Leftrightarrow v, w \in R \wedge (\forall t \in F) vw \neq f(t)\}. \tag{15}$$

By (15) and (12) we obtain

Proposition 3.1. *If $v, w \in R$, then $vw \notin R$ if and only if there exists a unique $t \in R$ such that $vw = f(t)$.*

A corollary of Proposition 3.1, by (4) is the following

Proposition 3.2. $\mathbf{R} = (R, \bullet)$ *is a groupoid.*

In what follows we will give the proofs of the main results, expressed by four theorems in Section 1.

Proof of Theorem 1: By induction on lengths of elements of R it can be easily obtained that B generates $\mathbf{R}$. It is also clear that if $b \in B$ then $R \setminus \{b\}$ is a subgroupoid of $\mathbf{R}$. This proves the part (i) of Theorem 1.

Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}^f$, $\lambda : B \to G$ be an arbitrary mapping and $\varphi : \mathbf{F} \to \mathbf{G}$ be the homomorphism which extends λ. Then, the restriction $\psi = \varphi | R$ of φ on R is a homomorphism from $\mathbf{R}$ into $\mathbf{G}$ which extends λ. Namely, let $v, w \in R$ be such that $vw \notin R$. By Proposition 3.1, there exists a unique $t \in R$ such that $vw = f(t)$, i.e. $v = f_1(t)$, $w = f_2(t)$, where $f = f_1 f_2$. Then

$$\begin{aligned} \psi(v \bullet w) &= \varphi(t) = f^{\mathbf{G}}(\varphi(t)) = \big(f_1^{\mathbf{G}}(\varphi(t))\big(f_2^{\mathbf{G}}(\varphi(t)\big) = \varphi(f_1(t))\varphi(f_2(t)) \\ &= \varphi(v)\varphi(w) = \psi(v)\psi(w), \end{aligned}$$

where Proposition 2.3 was used. According to the last relation and Proposition 3.2 the proof of Theorem 1 follows. □

Proof of Theorem 2: Below we assume that f is reduced and if $t \in R$, $h \in E$, then we write $h^{\bullet}(t)$ instead of $h^{\mathbf{R}}(t)$.

First, we will prove the following result: If $t \in R$ and g is a proper part of f, then

$$g^{\bullet}(t) = g(t) \tag{16}$$

for any $t \in R$.

Certainly, (16) holds in the case $g = e$. Assume that the equality $h^{\bullet}(t) = h(t)$ holds for any proper part h of f such that $|h| \leq k$, and there exists a proper part g of f such that $|g| = k + 1$ and $g^{\bullet}(u) \neq g(u)$ for some $u \in R$. Then, if $g = g_1 g_2$, we have

$$g^{\bullet}(u) = g_1^{\bullet}(u) \bullet g_2^{\bullet}(u) = g_1(u) \bullet g_2(u) \neq g_1(u)g_2(u) = g(u).$$

Hence, by (15), there exists a $t \in R$ such that $g_1(u) \bullet g_2(u) = t$, and $g_1(u)g_2(u) = f(t)$. In regard to (1) this implies $g_1(u) = f_1(t)$, $g_2(u) = f_2(t)$, where $f = f_1 f_2$.

The fact that g is a proper part of f implies that $|f| > |g|$ and, by virtue of (11), we have $|g||u| = |f||t|$, which gives $|u| > |t|$. In regard to Proposition 2.1, there exists a unique pair $(h_1, h_2) \in E^2$ such that $f_1 = g_1(h_1)$, $f_2 = g_2(h_2)$, $h_1(t) = u = h_2(t)$.

From the last two relations, by (12) we obtain $h_1 = h_2 (= h)$ and, therefore, we have $h \neq e$, $g \neq e$ and

$$f = f_1 f_2 = (g_1(h))(g_2(h)) = (g_1 g_2)(h) = g(h).$$

But, in view of (2), the last relation is impossible because f is reduced and so (16) follows by contradiction.

Furthermore, if $t \in R$, then by (16) we have

$$f^{\bullet}(t) = (f_1^{\bullet}(t)) \bullet (f_2^{\bullet}(t)) = f_1(t) \bullet f_2(t) = t,$$

which means that $\mathbf{R} \in \mathcal{V}^f$. According to this and Theorem 1 we furnish the proof of Theorem 2. □

Proof of Theorem 3: In what follows we will deal with $f = (e^m)^n$.

If $m = 1$ or $n = 1$, then by Proposition 2.2 f is reduced and the conclusion follows by Theorem 2.

It remains to show the following implications:

$$2 \leq m \leq n \quad \Rightarrow \quad \mathbf{R} \in \mathcal{V}^f, \quad 2 \leq n < m \quad \Rightarrow \quad \mathbf{R} \notin \mathcal{V}^f.$$

Below, $u_{\bullet}^k$ denotes the k-th power of u in $\mathbf{R}$, i.e.

$$u_{\bullet}^1 = u, \quad u_{\bullet}^{k+1} = u_{\bullet}^k \bullet u.$$

Thus, if $m = n = 2$

$$(u_{\bullet}^m)_{\bullet}^n = (u \bullet u) \bullet (u \bullet u) = \begin{cases} u^2 \bullet u^2 = u, & u^2 \in R \\ t \bullet t = t^2 = u, & u = t^2. \end{cases}$$

Also, if $m = 2 < n$, then

$$(u_{\bullet}^m)_{\bullet}^n = (u_{\bullet}^2)_{\bullet}^n = (u^2)_{\bullet}^n = (u^2)^n = u.$$

In the same way, by straightward computations, one can show in the general case $2 \leq m \leq n$, the equality:

$$(t_{\bullet}^m)_{\bullet}^n = t,$$

for any $t \in R$, i.e. $\mathbf{R} \in \mathcal{V}^f$. All this, together with Theorem 1, implies that $\mathbf{R}$ is a free groupoid in $\mathcal{V}^f$, with the basis B.

To complete the proof of Theorem 3 we will show that if $2 \leq n < m$, then $\mathbf{R} \notin \mathcal{V}^f$.

First, if $b \in B$, then $b_{\bullet}^k = b^k$ for any $k \geq 1$, and thus $t = b^m \in R$. Then

$$t_{\bullet}^n = (b^m)_{\bullet}^n = b,$$

and therefore

$$t_{\bullet}^m = b\underline{tm - n},$$

where the right side is defined by (6). Now we have

$$(t_{\bullet}^m)_{\bullet}^n = (b\underline{tm - n})^n \neq b^m = t,$$

and this implies that $\mathbf{R} \notin \mathcal{V}^f$, which completes the proof of Theorem 3. □

Proof of Theorem 4: Assume that $2 \leq n < m$, $f = (e^m)^n$ and that $\mathbf{S}$ and $*$ are defined by (7) and (8). The idea of these definitions has its origin in the following property. If $\mathbf{G} = (G, \cdot) \in \mathcal{V}^f$, then

$$(a^m)^m = (a^m)^n \underline{a^m m - n} = a\underline{a^m m - n},$$

for each $a \in G$.

To show that $\mathbf{S} \in \mathcal{V}^f$, assume that $u \in S$. Then

$$u_*^i = u^i$$

for each i, $1 \leq i < n$, and

$$u_*^n = \begin{cases} u^n, & (\forall t \in F, p \geq 0) u \neq t^{<p+1>} \\ t^{<p>}, & u = t^{<p+1>}. \end{cases}$$

Following the same argumentation we obtain

$$u_*^m = \begin{cases} u^m, & (\forall t \in F, p \geq 0) u \neq t^{<p+1>} \\ t^{<p>}\underline{t^{<p+1>}m - n}, & u = t^{<p+1>}. \end{cases}$$

Therefore

$$(u_*^m)_*^n = \begin{cases} (u^{<1>})_*^n = u, & (\forall t \in F, p \geq 0) u \neq t^{<p+1>} \\ t^{<p+1>} = u, & u = t^{<p+1>}, \end{cases}$$

and this implies that $\mathbf{S} \in \mathcal{V}^f$.

Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}^f$ and $\lambda : B \to G$ be a mapping. Let $\varphi : F \to G$ be the homomorphism which extends λ. We will show that the restriction $\psi = \varphi \mid S$ of φ on S is a homomorphism from $\mathbf{S}$ into $\mathbf{G}$.

It is enough to show that

$$\varphi(v * w) = \varphi(v)\varphi(w)$$

when $vw \notin S$. In that case we have

$$v = w^{n-1}, \quad w = t^{<p+1>}, \quad v * w = t^{<p>},$$

so that by (14)

$$\begin{aligned}
\varphi(v)\varphi(w) &= \varphi((t^{<p+1>})^{n-1})\varphi(t^{<p+1>}) \\
&= \varphi(t^{<p+1>})^{n-1}\varphi(t)^{<p+1>} \\
&= (\varphi(t)^{<p+1>})^{n-1}\varphi(t)^{<p+1>} \\
&= (\varphi(t)^{(p+1)})^{n-1}\varphi(t)^{(p+1)} \\
&= (\varphi(t)^{(p+1)})^{n} = ((\varphi(t)^{(p)})^{m})^{n} \\
&= \varphi(t)^{(p)} = \varphi(t)^{<p>} = \varphi(t^{<p>}) \\
&= \varphi(v * w).
\end{aligned}$$

Thus, $\mathbf{S}$ is a free object in $\mathcal{V}^f$ with the basis B, and this completes the proof. □

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