Математички Билтен
23 (XLIX)
1999 (19–24)
Скопје, Македонија

ISSN 0351-336X

CONGRUENCES ON (n,m)-GROUPS

Biljana Janeva

Abstract

In this paper we give a generalization of some definitions and properties of congruences on n-groups given in [2]. Congruences on (n,m)-groups are defined as congruences of the corresponding component algebra, and as kernel of a homomorphism, and a connection between these two definitions is given. Also, it is shown that for each congruence of an (n,m)-group $\boldsymbol{Q}$ there exists an invariant subgroup of its universal covering group $\boldsymbol{Q}^{\vee}$ of $\boldsymbol{Q}$, that is a subset of Q_{m+p}, where $m+p=sk$, $k=n-m>0$. Conversely, for each invariant subgroup K of $\boldsymbol{Q}^{\vee}$, which is a subset of $\boldsymbol{Q}_{m+p}$ and satisfies the condition

$$x_j y_j^{-1} \in K,\ j=\overline{1,n}\ \&\ x_1\cdots x_n = a_1\cdots a_m, y_1\cdots y_n = \\ = b_1\cdots b_m \Rightarrow a_i b_i^{-1} \in K,$$

for all $i=\overline{1,m}$, there exists a congruence α on the (n,m)-group $\boldsymbol{Q}$, such that the corresponding invariant subgroup of $\boldsymbol{Q}^{\vee}$ is exactly K.

1. Preliminary definitions and results

Let Q be a nonempty set. Denote by Q^r, r is a positive integer, the r-th Cartesian power of Q. Instead of denoting the elements of Q^r by $(a_1,\ldots,a_r)$ we will use the notation $a_1\cdots a_r$, or a_1^r. In this way we can identify the set Q^r with the subset $\{a_1\cdots a_r \mid a_\nu \in Q\}$ of the free semigroup Q^+ with a basis Q. The element $a_i\cdots a_j \in Q^+$ will be denoted by a_i^j, meaning the empty symbol when $j<i$, i.e. the unity of $Q^* = Q^+ \cup \{1\}, 1 \notin Q^+$.

Let $m, n, n - m = k > 0$ be positive integers, and $f : Q^n \to Q^m$ a mapping. Then we say that $\boldsymbol{Q} = (Q; f)$ is an (n,m)-groupoid, and f is an (n,m)-operation on Q. If, moreover, $\boldsymbol{Q}$ satisfies the condition

$$f(f(x_1^n)x_{n+1}^{2k+m}) = f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{2k+m}),$$

for each $1 \leq i \leq k$, $x_\nu \in Q$, then we say that f is an associative (n,m)-operation. We say that the ordered pair $(Q; f)$, where f [1] is an associative (n,m)-operation, is an *(n,m)-semigroup.*

If $\boldsymbol{Q} = (Q; [\ ])$ is an (n,m)-semigroup, then the semigroup $\boldsymbol{Q}^\wedge$ given by the presentation

$$\boldsymbol{Q}^\wedge = \langle Q \mid \{(a_1^n, b_1^m); [a_1^n] = b_1^m\}\rangle$$

in the class of all semigroups, is said to be *the universal covering semigroup* of $\boldsymbol{Q}$.

The carrier $Q^\wedge$ of $\boldsymbol{Q}^\wedge$ is a disjoint union of the form

$$Q^\wedge = Q \cup Q^2 \cup \cdots \cup Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1},$$

where $Q_{m+i} = Q^{m+i}/\beta$ such that β is the congruence on $\boldsymbol{Q}^\wedge$ induced by the defining relations of its presentation ([1]). We denote by $Q^\vee$ the subset $Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1}$ of $Q^\wedge$. $Q^\vee$ is an ideal of $\boldsymbol{Q}^\wedge$ ([1]).

In this case we have the following property

$$i \leq m, x_\nu \in Q \Rightarrow (x_1 \cdots x_i = y_1 \cdots y_i \Rightarrow x_\nu = y_\nu), \nu = \overline{1,i}.$$

For each (n,m)-groupoid we can associate an algebra with m n-ary operations defined by $[a_1^n]_i = b_i$ iff $[a_1^n] = b_1^m$, where $i = \overline{1,m}$, and $a_\nu, b_\lambda \in Q$ [2]. Then $(Q; [\ ]_1, \ldots, [\ ]_m)$ is called a *component algebra* of $\boldsymbol{Q}$.

Let $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ be two (n,m)-semigroups and $\varphi : Q \to Q'$ a mapping. We say that φ is an *(n,m)-homomorphism* if it is a homomorphism between their corresponding component algebras. We can define a mapping $\varphi^\wedge$ between the corresponding universal covering semigroups by

$$\varphi^\wedge(x_1^i) = \varphi(x_1) \cdots \varphi(x_i). \tag{1.1}$$

Then $\varphi^\wedge$ is a *homomorphism induced by φ.*

We give, bellow, some connections between a homomorphism of (n,m)-semigroups and the induced one of their universal covering semigroups.

---

[1] Further on we will denote an (n,m)-operation by [ ].

[2] Further on we will assume that $a_\nu, b_\lambda \in Q$.

Prop 1.1. *Let $\varphi : Q \to Q'$ be a homomorphism of (n,m)-semigroups and $\varphi^\wedge$ the induced homomorphism defined by* (1.1). *Then $\varphi^\wedge$ is the unique homomorphique extension of φ and φ is surjective iff $\varphi^\wedge$ is surjective as well* ([1]).

Let $\boldsymbol{Q}$=$(Q;[\ ])$ be an (n,m)-semigroup, such that for all $a_\nu, b_\lambda \in Q$, there exist $x_i, b_i \in Q$, such that

$$[a_1^k x_1^m] = b_1^m = [y_1^m a_1^k].$$

Then we say that $\boldsymbol{Q}$ is an (n,m)-*group*.

In this case, when $\boldsymbol{Q}$ is an (n,m)-group, $\boldsymbol{Q}^\wedge$ is the universal covering semigroup and

$$\boldsymbol{Q}^\vee = Q^m \cup Q_{m+1} \cup \cdots \cup Q_{m+k-1}$$

is a group, called *universal covering group of* $\boldsymbol{Q}$ ([1]).

For each $a \in Q$, $\boldsymbol{Q}^\vee$ has the form

$$Q^\vee = Q_m \cup aQ_m \cup \cdots \cup a_{k-1}Q_m,$$

and the unity $\mathbf{1}$ of $\boldsymbol{Q}^\vee$ is an element of Q_{m+p}, where $m+p < m+k$, and $m+p = sk$. Thus, Q can be considered as a subset of Q_{m+p+1}.

Prop 1.2. *Let $\boldsymbol{Q} = (Q;[\ ])$ and $\boldsymbol{Q}' = (Q';[\ ]')$ be (n,m)-groups, and $\varphi : Q \to Q'$ a homomorphism. Then there exists a unique extension $\varphi^\vee : Q^\vee \to Q'^\vee$ of φ, defined by*

$$\varphi^\vee(x_1 \cdots x_{m+i}) = \varphi(x_1) \cdots \varphi(x_{m+i}),$$

and φ is surjective (injective) iff $\varphi^\vee$ is surjective (injective) as well ([1]).

2. Congruences on (n,m)-semigroups

Let α be an equivalence relation on Q. We say that α is a *congruence* on $\boldsymbol{Q}$ if for $i = \overline{1,n}$, we have

$$a_i \alpha b_i \Rightarrow [a_1^n]_j \alpha [b_1^n]_j, j = \overline{1,m},$$

i.e. if α is a congruence on the corresponding component algebra of $\boldsymbol{Q}$.

Let α be a congruence on an (n,m)-semigroup $\boldsymbol{Q}$, and $\varphi = \operatorname{nat}\alpha : Q \to Q/\alpha$ the natural homomorphism. Then $\varphi^\wedge : Q^\wedge \to (Q/\alpha)^\wedge$ is an epimorphism, and $\alpha^\wedge = ker\varphi^\wedge$ a congruence on $\boldsymbol{Q}^\wedge$. Thus

Prop 2.1. $(Q/\alpha)^\wedge \cong Q^\wedge/\alpha^\wedge$.

Using the properties of the universal covering semigroup of an (n,m)-semigroup and the definition of congruences of (n,m)-semigroups, we will give below some connections between congruences of the given (n,m)-semigroup and its universal covering semigroup.

Prop 2.2. *Let β be a congruence on $\boldsymbol{Q}^{\wedge}$ with the property*

$$x_j\beta y_j, j=\overline{1,n}\& x_1^n=a_1^m, y_1^n=b_1^m \Rightarrow a_i\beta b_i, i=\overline{1,m}. \tag{2.1}$$

Then $\alpha=\beta_{/Q}$ is a congruence on $\boldsymbol{Q}$, such that $\alpha^{\wedge}\subseteq\beta$.

Prop 2.3. *Let β be a congruence of $\boldsymbol{Q}^{\wedge}$, such that satisfies* (2.1) *and*

$$x_\nu, y_\nu \in Q, i,j<m+k \Rightarrow (x_1^i\beta y_1^j \Rightarrow i=j). \tag{2.2}$$

Then $\alpha=\beta_{/Q}$ is a congruence on $\boldsymbol{Q}$, such that $\alpha^{\wedge}=\beta$.

3. Congruences on (n,m)-groups

Let $\boldsymbol{Q}$ be an (n,m)-group, and α a congruence on $\boldsymbol{Q}$. Define a relation $\alpha^{\vee}$ on $\boldsymbol{Q}^{\vee}$ by

$$a^ix_1^m\alpha^{\vee}a^iy_1^m \Leftrightarrow x_i\alpha y_i, i=\overline{1,m}. \tag{3.1}$$

Then

Prop 3.1. (*i*) *$\alpha^{\vee}$ does not depend on the choice of a;* (*ii*) *$\alpha^{\vee}$ is a congruence on $\boldsymbol{Q}^{\vee}$, such that $\alpha^{\vee}_{/Q}=\alpha$.*

Thus, for each congruence α on an (n,m)-group, there is a congruence, namely $\alpha^{\vee}$, of its universal covering group, such that $\alpha^{\vee}_{/Q}=\alpha$.

To be able to establish connections between the congruences of an (n,m)-group and its universal covering group, let us first give some properties of the congruence $\alpha^{\vee}$ on $\boldsymbol{Q}^{\vee}$ induced by a given congruence α of the given (n,m)-group $\boldsymbol{Q}$.

Prop 3.2. *Let α be a congruence on the (n,m)-group $\boldsymbol{Q}$, $\varphi=$ nat α, $\varphi^{\vee}:Q^{\vee}\to(Q/\alpha)^{\vee}$, and $\overline{\alpha}=ker\varphi^{\vee}$. Then $\overline{\alpha}=\alpha^{\vee}$.*

Prop 3.3. *Let α be a congruence on the (n,m)-group $\boldsymbol{Q}$. Then $\alpha^{\vee}$ satisfies the following conditions*

$$(i)\ x_j\alpha^{\vee}y_j\ \& x_1^n=a_1^m, y_1^n=b_1^m \Rightarrow a_i\alpha^{\vee}b_i, i=\overline{1,m}; \tag{3.2}$$

(*ii*) *The invariant subgroup K, induced by $\alpha^{\vee}$ is a subset of Q_{m+p}, where $m+p=sk$, $0\le p\le k-1$.*

Prop 3.4. *Let β be a congruence on $\boldsymbol{Q}^{\vee}$, $\alpha=\beta_{/Q}$ and β satisfy* (3.2). *Then α is a congruence on $\boldsymbol{Q}$, such that $\alpha^{\vee}\subseteq\beta$.*

The next proposition establishes connections under which the restriction $\alpha=\beta_{/Q}$ of the congruence β of the universal covering group is such that $\alpha^{\vee}=\beta$.

Prop 3.5. *Let β be a congruence on $\boldsymbol{Q}^{\vee}$, $\alpha=\beta_{/Q}$, β satisfies* (3.2) *and*

$$0\le i,j<k\,\&\,x_1^{m+i}\,\beta\,y_1^{m+j} \Rightarrow i=j.$$

Then α is a congruence on $\boldsymbol{Q}$, such that $\alpha^{\vee} = \beta$.

Prop 3.6. *For each congruence α on the (n, m)-group $\boldsymbol{Q}$, there exists an invariant subgroup $K \subseteq Q_{m+p}$, such that*

$$x_j y_j^{-1} \in K,\ j = \overline{1, n} \,\&\, x_1^n = a_1^m,\ y_1^n = b_1^m \Rightarrow a_i b_i^{-1} \in K,\ i = \overline{1, m}. \qquad (3.3)$$

Conversely, for each invariant subgroup K of $\boldsymbol{Q}^{\vee}$, such that $K \subseteq Q_{m+p}$ and (3.3) is satisfied, there exists a congruence α on the (n, m)-group $\boldsymbol{Q}$, such that the invariant subgroup induced by $\alpha^{\vee}$ is exactly K.

Thus Prop 3.6 is a characterization of the congruences of an (n, m)-group through invariant subgroups of its universal covering group.

As a corollary of Prop 3.6 we obtain the following

Prop 3.7. *The lattice of congruences of an (n, m)-group is a modular one and is isomorphic to a sublattice of the lattice of invariant subgroups of $\boldsymbol{Q}^{\vee}$.*

References

[1] Čupona Ǵ., Celakoski N., Markovski S., and Dimovski D.: *Vector Valued Groupoids, Semigroups and Groups*, Vector Valued Semigroups and Groups, Maced. Acad. Sci. Art., Skopje, (1988), 1–78

[2] Janeva B.: *Congruences on n-Groups*, Bull. Soc. Math. Mac., Skopje **19** (XLV) (1995), 85-90.

[3] Markovski S. and Janeva B.: *Post and Hosszu-Gluskin Theorem for Vector Valued Groups*. Proc. Conf. "Algebra and Logic", Sarajevo (1987) 77-88.

[4] Post E.L.: *Polyadic Groups*, Trans, Amer. Math. Soc. **48** (1940), 208-250.

КОНГРУЕНЦИИ НА (n,m)-ГРУПИ

Билјана Јанева

Резиме

Во овој труд е дадено обопштување на некои дефиниции и својства на конгруенции на n-групи ([2]). Конгруенциите на (n,m)-групи се дефинирани како конгруенции на соодветната компонетна алгебра и како јадро на хомоморфизам. Исто така дадена е и врската меѓу овие две дефиниции. Покрај тоа, покажано е дека за секоја конгруенција на (n,m)-група $\boldsymbol{Q}$ постои нормална подгрупа од нејзината универзална покривачка група $\boldsymbol{Q}^\vee$ која е подмножество од Q_{m+p}, каде $m+p=sk$, $k=n-m>0$, како и дека за секоја нормална подгрупа K од $Q^\vee$ која е подмножество од Q_{m+p} и го задоволува условот

$$x_j y_j^{-1} \in K\ j=\overline{1,n}\ \&\ x_1\cdots x_n = a_1\cdots a_m, y_1\cdots y_n = b_1\cdots b_m$$
$$\Rightarrow a_i b_i^{-1}\in K,\ i=\overline{1,m},$$

постои конгруенција α на (n,m)-групата $\boldsymbol{Q}$, така што соодветната нормална подгрупа од $Q^\vee$ е точно K.

Faculty of Natural Sciences and Mathematics
P. O. Box 162
1000 Skopje
Macedonia