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INJECTIVE VECTOR VALUED SEMIGROUPS

Dončo Dimovski
Institute of Mathematics, University of Skopje
P.O. Box 162, 91000 Skopje, Macedonia
e-mail: *donco@iunona.pmf.ukim.edu.mk*

Ǵorǵi Čupona
Macedonian Academy of Science and Arts
91000 Skopje, Macedonia

Abstract

In this paper we introduce the notion of injective vector valued semigroups. Then we show that the class of injective vector valued semigroups is larger than the class of free vector valued semigroups, and give a necessary and sufficient condition for injective vector valued semigroups to be free vector valued semigroups. All this gives a way for checking if a given vector valued semigroup is free.

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1. Preliminary notions

We start with several notions and definitions.

The set of positive integers is $\mathbf{N} = \{1, 2, 3, \ldots\}$, and the set of the first m positive integers is $\mathbf{N}_m = \{1, 2, 3, \ldots, m\}$. Moreover, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and $\mathbf{N}_{k,0} = \mathbf{N}_k \cup \{0\}$.

For a given set $Q \neq \emptyset$, and $t \in \mathbf{N}$, let Q^t be the cartesian product of t copies of Q. If $\mathbf{x} = (a_1, a_2, \ldots, a_t) \in Q^t$, then, for short, we write

$\mathbf{x} = a_1^t$, and moreover we identify $\mathbf{x}$ with the word $a_1a_2\dots a_t$. For such an $\mathbf{x}$ we say that its length $|\mathbf{x}|$ is t, and its contents is the set $cn(\mathbf{x}) = \{a|a = a_i$ for some a_i in the word $\mathbf{x} = a_1a_2\dots a_t\}$. Let Q^+ be the union of all the cartesian products Q^t, for $t \in \mathbf{N}$.

For the rest of the paper, let $n, m, k \in \mathbf{N}$, $n = m + k$ be given.

For a set $Q \neq \emptyset$, let $Q^{m,k} = \{\mathbf{x}|\mathbf{x} \in Q^+, |\mathbf{x}| = m + sk, s \in \mathbf{N}\}$.

We recall the following definitions, given in [1, 2]. A pair $\mathbf{Q} = (Q, f)$ for a map $f : Q^n \to Q^m$ is called an (n,m)-*groupoid*, i.e. a *vector valued groupoid*, written in short as VVG, and the map f is called (n,m)-operation. A VVG, i.e. an (n,m)-groupoid $\mathbf{Q} = (Q, f)$ is called (n,m)-semigroup, i.e. *vector valued semigroup*, written in short VVS, if $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{n+k}, \mathbf{y}, \mathbf{v} \in Q^n$, i.e. if the associative law holds for the (n,m)-operation f. Because of the associative law, the operation f can be extended to an operation, denoted by the same letter, $f : Q^{m,k} \to Q^m$ such that for each $\mathbf{xyz}\in Q^{m,k}$, and $\mathbf{y} \in Q^{m,k}, f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{xyz})$. The notion of (n,m)-semigroup is introduced in [5] and is examined in more details in [1], while a convenient construction of free (n,m)-semigroups is given in [4]. It is obvious that a (2,1)-semigroup is a usual semigroup, and that an $(n,1)$-semigroup is an n-semigroup.

A VVG (Q, f) can be considered as an algebra with m n-ary operations $f_j : Q^n \to Q$, where $j \in \mathbf{N}_m$. These operations can be extended to an infinite family of operations $f_{j,s} : Q^{m+sk} \to Q$ for $s \in \mathbf{N}$, where for a given s, there are more than one operation $f_{j,s}$. In the semigroup case, for each $s \in \mathbf{N}$, there is only one operation $f_{j,s} : Q^{m+sk} \to Q$, whose union is a map $f_j : Q^{m,k} \to Q^m$. The above discussion shows that all the notions, (such as generating set, (n,m)-subsemigroup, homomorphism, a free (n,m)-semigroup), in a variety of algebras hold in the variety of (n,m)-semigroups.

Let $\mathbf{Q} = (Q, f)$ be a VVS, and let $\mathbf{x} \in Q^{m,k}$. We say that $\mathbf{x}$ is: *reducible*, if $\mathbf{x} = \mathbf{x}'\mathbf{y}\mathbf{x}''$, where $\mathbf{y} = f(\mathbf{z})$ for some $\mathbf{z} \in Q^{m,k}$, and *irreducible*, otherwise. Usually, we denote the set of all the irreducible words by $R(\mathbf{Q})$.

For a given VVS $\mathbf{Q} = (Q, f)$, let $P(\mathbf{Q}) = Q\setminus \cup\, cnf(\mathbf{x})$, where the union is over all $\mathbf{x} \in Q^{m,k}$. The elements of $P(\mathbf{Q})$ are called *prime elements* of Q.

2. Free vector valued semigroups

Below we will give a construction of a canonical form of a free VVS, slightly different from the construction given in [4]. Let $B \neq \emptyset$, and for each

$j \in \mathbf{N}_m$, let ρ^j be a symbol interpreted as an n-ary functional symbol. Let $\mathbf{F} = (F; \rho^1, \rho^2, \ldots, \rho^m)$ be the free algebra with the basis B, of type $\Omega = \{\rho_r^j | j \in \mathbf{N}_m, r \geq 1\}$, where $\rho_r^j \in \Omega_{m+rk}$. So, the elements of F are all the elements of B, and all the elements of the form $\rho_r^j(\mathbf{x})$, for $\mathbf{x} \in F^{m+rk}, r \geq 1$. By choosing different letters, if necessary, for the elements of B, we will have that no element of B is of the form $\rho_r^j(\mathbf{x})$. From now on we are not going to write the lower index. Two elements $\rho^j(\mathbf{x}), \rho^t(\mathbf{y}) \in F$ are equal if and only if $j = t$, and $\mathbf{x}=\mathbf{y}$. For each $u \in F$ we define the norm of u, denoted by $|u|$, to be 1 for $u \in B$, and by induction $|\rho^j(u_1^{m+rk})| = \sum_{t=1}^{m+rk} |u_t|$. Thus, $|\rho^j(u_1^{m+rk})|$ is the number of appearances of elements from B in $\rho^j(u_1^{m+rk})$. We define also the norm of an element $\mathbf{x} \in F^+, \mathbf{x} = u_1^s$, to be $|u_1^s| = \sum_{t=1}^{s} |u_t|$. We say that an element $u \in F$ is *reducible* if it has the form $u = \rho^j(\mathbf{xyz})$, such that $\mathbf{xyz} \in F^{m,k}, \mathbf{y} \in F^m, \mathbf{y} = y_1^m$, and $y_j = \rho^j(\mathbf{w})$ for some $\mathbf{w}$. Otherwise, we say that $u \in F$ is *irreducible*. Let $F(B)$ be the set of all the irreducible elements in F. For an element $\mathbf{x} = u_1^{m+rk} \in F^+$, we say that is *irreducible* if all the u_t are in $F(B)$, and $\rho^1(u_1^{m+rk})$ is in $F(B)$, which is equivalent to $\rho^j(u_1^{m+rk}) \in F(B)$, for each $j \in \mathbf{N}_m$. For each $j \in \mathbf{N}_m$, we define a map $f_j : F(B)^{m,k} \to F(B)$ as follows:

If $\mathbf{u} \in F(B)^{m,k}$ is such that $\rho^1(\mathbf{u}) \in F(B)$, then $f_j(\mathbf{u}) = \rho^j(\mathbf{u})$.

Let $\mathbf{u} = u_1^{m+rk} \in F(B)^{m,k}$ be such that $\rho^1(u_1^{m+rk}) \notin F(B)$. Then, $\rho^t(u_1^{m+rk}) \notin F(B)$ for every $t \in \mathbf{N}_m$, and there is $p, 0 \leq p \leq rk$, such that $u_{p+i} = \rho^i(\mathbf{v})$, for some $\mathbf{v} \in F(B)^{m,k}$ and each $i \in \mathbf{N}_m$. Let p be the smallest such number. Then, by induction on the norm of elements in $F(B)^+$, we define:

$$f_j(\mathbf{u}) = f_j(u_1^{m+rk}) = f_j(u_1^p \mathbf{v} u_{p+m+1}^{m+rk}).$$

Next, we define $f : F(B)^{m,k} \to F(B)^m$, by $f(\mathbf{x}) = f_1(\mathbf{x}) f_2(\mathbf{x}) \ldots f_m(\mathbf{x})$. The proof of the following theorem, is by induction on the norm and using the above definition.

Theorem 2.1. $\mathbf{F}(B) = (F(B), f)$ *is a free* (n, m)*-semigroup with basis* B.

Further, we will state several properties of $\mathbf{F}(B)$, whose proof follows from the definition of $\mathbf{F}(B)$ using induction on the norm. These properties led us to the notion of injective VVS.

Theorem 2.2.

(1) The set of prime elements in $\mathbf{F}(B)$ *is* B, *i.e.* $P(\mathbf{F}(B)) = B$;

(2) *The set* $R = R(\mathbf{F}(B))$ *of all the irreducible elements in* $\mathbf{F}(B)$ *consists of all* $\mathbf{x} \in F(B)^{m,k}$ *such that* $\rho^1(\mathbf{x}) \in F(B)$;

(3) *If* $\mathbf{x} \in R$, *then* $\rho^j(\mathbf{x}) \in F(B)$ *for each* $j \in \mathbf{N}_m$;

(4) *For each* $\mathbf{x} \in F(B)^{m,k}$ *there exists* $\mathbf{y} \in R$, *such that* $f(\mathbf{x}) = f(\mathbf{y})$, *and for each* $j \in \mathbf{N}_m, f_j(\mathbf{x}) = \rho^j(\mathbf{y})$;

(5) *If* $\mathbf{x} \in R = R(\mathbf{F}(B))$ *then for each* $j \in \mathbf{N}_m, |\rho^j(\mathbf{x})| = |\mathbf{x}|$;

(6) *If* $f_j(\mathbf{x}) = f_i(\mathbf{y})$ *for some* $\mathbf{x}, \mathbf{y} \in F(B)^{m,k}$, *then* $i = j$ *and for each* $t \in \mathbf{N}_m, f_t(\mathbf{x}) = f_t(\mathbf{y})$.

(7) *If* $\mathbf{x}, \mathbf{y} \in R$, *then* $f(\mathbf{x}) = f(\mathbf{y})$ *implies that* $\mathbf{x}=\mathbf{y}$.

(8) $F(B) = B \cup im\ f_1 \cup im\ f_2 \cup \ldots \cup im\ f_m$, *where the union is disjoint.*

At the end of this part we note that in the case $n = 2, m = 1, k = 1$: $\mathbf{F}(B) = B \cup \{\rho(\mathbf{x}) | \mathbf{x} \in B^+ \setminus B\}$, i.e. after replacing $\rho(\mathbf{x})$ by $\mathbf{x}$, we obtain the well known description of the free semigroup $(B^+, \cdot)$ with the basis B.

3. Injective vector valued semigroups

The above properties of free VVS led to a class of VVS called injective VVS. Let $\mathbf{Q} = (Q, f)$ be an (n,m)-semigroup, $n = m + k$ and let $R(\mathbf{Q})$ be its set of irreducible words. We call $\mathbf{Q}$ *injective* VVS if:

(Inj.1.) If $f_j(\mathbf{a}) = f_i(\mathbf{b})$ for some $i, j \in \mathbf{N}_m$ and $\mathbf{a}, \mathbf{b} \in Q^{m,k}$, then $i = j$ and $f(\mathbf{a}) = f(\mathbf{b})$,

(Inj.2.) For each $j \in \mathbf{N}_m$, if $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$, and $f_j(\mathbf{x}) = f_j(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of each f_j on $R(\mathbf{Q})$ is an injection.

The conditions *(Inj.1.)* and *(Inj.2.)* are equivalent to the conditions *(Inj.1.)* and *(Inj.2'.)*., where:

(Inj.2'.) If $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$, and $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of f on $R(\mathbf{Q})$ is an injection.

Theorem 2.2 implies

Proposition 3.1. *A free VVS is an injective VVS.*

For a binary, i.e. a (2,1)-semigroup, $(Q,\cdot)$, an element is prime if it is not a product of two elements, and a word of elements, i.e. $\mathbf{x} \in Q^+$ is irreducible if it is a word of prime elements. A semigroup $(Q,\cdot)$ is injective if and only if for irreducible $a_1^k, b_1^t \in Q^+, a_1 \cdot a_2 \cdot \ldots \cdot a_k = b_1 \cdot b_2 \cdot \ldots \cdot b_t$ implies $k = t$ and $a_1^k = b_1^k$ in Q^+.

It is easy to check that a semigroup is free if and only if it is injective and is generated by the set of prime elements.

The following main theorem in this paper is a generalization of the previous fact.

Theorem 3.2. *A VVS is free if and only if it is injective and is generated by the set of prime elements.*

Proof. Proposition 3.1 and Theorem 2.2, imply that a free VVS is injective and is generated by the set of prime elements.

Conversely, let $\mathbf{H} = (H, h)$ be an injective (n,m)-semigroup generated by its set of prime elements $P(\mathbf{H})$=P. Let $R(\mathbf{H}) = R \subseteq H^{m,k}$ be the set of irreducible words. Let $B = P$, and let $\mathbf{F}(B) = (F(B), f)$ be the free (n,m)-semigroup as constructed in Theorem 2.1. Let $\xi : B \to H$ be defined by $\xi(b) = b$, and let $\varphi : F(B) \to H$ be the homomorphism extending ξ. We extend the map φ to a map $\psi : F(B)^{m,k} \to H^{m,k}$, by: $\psi(x_1^{m+sk}) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_{m+sk})$. Then, for each $a = \rho^j(\mathbf{u}) \in F(B), \varphi(a) = h_j(\psi(\mathbf{u}))$. Since φ is an extension of ξ, which is in fact the identity map from $B = P$ to P, and since $\mathbf{H}$ is generated by $P = B$, it follows that the map φ is a surjection. In order to show that $\mathbf{H} = (H, h)$ is a free VVS, it is enough to show that the map φ is an injection.

First, by induction on the norm of the elements in $F(B)$ and $F(B)^{m,k}$ we will show that for each $t \in \mathbf{N}$:

(a) If $\mathbf{x} \in R = R(\mathbf{F}(B))$ and $|\mathbf{x}| \leq t$, then $\psi(\mathbf{x}) \in R(\mathbf{H})$; and

(b) If $u, v \in F(B)$ and $|u|, |v| \leq t$, then $\varphi(u) = \varphi(v)$ implies $u = v$.

For $t = 1$, the condition is trivially true, since the elements in R have norm bigger than 1. The condition (b) is true by the definition of φ, which is an extension of the identity map ξ.

Next, let (a) and (b) be true for each $t \leq q - 1$.

Prove first (a) for $t = q$.

Let $\mathbf{x} \in R = R(\mathbf{F}(B)), |\mathbf{x}| = q, \mathbf{x} = x_1^{m+sk}$, and let $\psi(\mathbf{x}) \notin R(\mathbf{H})$. Then, there is $0 \leq p \leq sk$, and $\mathbf{c} \in H^{m,k}$, such that $\varphi(x_{p+i}) = h_i(\mathbf{c})$, for each

$i \in \mathbf{N}_m$, i.e. $\psi(\mathbf{x}) = \psi(x_1^p)h(\mathbf{c})\psi(x_{p+m+1}^{m+sk})$. Since $\varphi(x_{p+i}) = h_i(\mathbf{c})$, for each $i \in \mathbf{N}_m$, it follows that $\varphi(x_{p+i}) \notin P = B$, and so, $x_{p+i} \notin B$ for each $i \in \mathbf{N}_m$. This implies that for each $i \in \mathbf{N}_m$, there is $\mathbf{v}_i \in R = R(\mathbf{F}(B))$, such that $x_{p+i} = \rho^j(\mathbf{v}_i)$, for some $j \in \mathbf{N}_m$. Then, $h_i(\mathbf{c}) = \varphi(x_{p+i}) = h_j(\psi(\mathbf{v}_i))$ and *(Inj.1.)* for $\mathbf{H}$, imply that j=i. Hence, for each $i \in \mathbf{N}_m$, $x_{p+i} = \rho^i(\mathbf{v}_i) \in F(B)$, and by Theorem 2.2.(5), $|\mathbf{v}_i| = |x_{p+i}| < |\mathbf{x}|$, i.e. $|\mathbf{v}_i| \leq q-1$.

Then, $h_i(\mathbf{c}) = \varphi(x_{p+i}) = \varphi(\rho^i(\mathbf{v}_i)) = h_i(\psi(\mathbf{v}_i))$, and *(Inj.1.)* imply that for each $i \in \mathbf{N}_m$, $h(\mathbf{c}) = h(\psi(\mathbf{v}_i))$. Further, since $|\mathbf{v}_i| \leq q-1$ and $\mathbf{v}_i \in R = R(\mathbf{F}(B))$, the induction hypothesis for (a), implies that $\psi(\mathbf{v}_i) \in R(\mathbf{H})$, for each $i \in \mathbf{N}_m$. Now, *(Inj.2.)* together with the equalities

$$h(\mathbf{c}) = h(\psi(\mathbf{v}_1)) = h(\psi(\mathbf{v}_2)) \ldots = h(\psi(\mathbf{v}_m)),$$

implies that $\psi(\mathbf{v}_1) = \psi(\mathbf{v}_2) \ldots = \psi(\mathbf{v}_m)$. Then, for $x_{p+1} = \rho^1(\mathbf{v}_1)$ and $y = \rho^1(\mathbf{v}_j)$ in $F(B)$, we have, $|x_{p+1}| \leq q-1$ and $|y| = |\mathbf{v}_j| \leq q-1$, which by the inductive hypothesis for (b) and the fact that $\varphi(x_{p+1}) = h_1(\psi(\mathbf{v}_1) = h_1(\psi(\mathbf{v}_j)) = \varphi(y)$, implies that $x_{p+1} = y$, i.e. $\rho^1(\mathbf{v}_1) = \rho^1(\mathbf{v}_j)$. The last equality shows that for each $j \in \mathbf{N}_m, \mathbf{v}_1 = \mathbf{v}_j = \mathbf{v} \in R(\mathbf{F}(B))$. This implies that $x_1^m = f(\mathbf{v})$, i.e. that $\mathbf{x} = x_1^p f(\mathbf{x}) x_{p+m+1}^{m+sk}$ is not irreducible, i.e. that $\mathbf{x} \notin R(\mathbf{F}(B))$.

Now we prove (b) for $t = q$.

Let $u, v \in \mathbf{F}(B)$, $u = \rho^i(\mathbf{x})$, $v = \rho^j(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in R(\mathbf{F}(B))$, $\mathbf{x} = x_1^{m+sk}$, $\mathbf{y} = y_1^{m+rk}$, $|\mathbf{x}| = |u| \leq q$, $|\mathbf{y}| = |v| \leq q$ and let $\varphi(u) = \varphi(v)$. Since $\mathbf{x}, \mathbf{y} \in R(\mathbf{F}(B))$ and $|\mathbf{x}| \leq q, |\mathbf{y}| \leq q$, the part (a) implies that $\psi(u), \psi(v) \in R(\mathbf{H})$. Further, since $h_i(\psi(\mathbf{x})) = \varphi(u) = \varphi(v) = h_j(\psi(\mathbf{y}))$, *(Inj.1.)* and *(Inj.2.)* for $\mathbf{H} = (H, h)$, imply that $i = j$, $h(\psi(\mathbf{x})) = h(\psi(\mathbf{y}))$, and $\psi(\mathbf{x}) = \psi(\mathbf{y})$ as words. But this implies that $s = r$ and for each $1 \leq p \leq m + sk$, $\varphi(x_p) = \varphi(y_p)$. Now, by the induction hypothesis, since $|x_p| \leq q-1$ and $|y_p| \leq q-1$, it follows that $x_p = y_p$. Hence, $\mathbf{x} = \mathbf{y}$, and so $u = v$.

A direct application of (b) implies that φ is an injection, and so it is an isomorphism. Hence, $\mathbf{H} = (H, h)$ is a free (n, m)-semigroup with a basis $P = B$. □

4. Examples

The following examples will show that the class of injective VVS is larger than the class of free VVS.

Example 4.1. Let $\mathbf{F} = (F(a), f)$ be the free (n,m)-semigroup generated by one element a, and let $m \geq 2$. Let $Q = F(a)\backslash\{a\}$, and let g, as an (n,m)-operation on Q, be the restriction of the (n,m)-operation f on $F(a)$. Then $\mathbf{Q} = (Q,g)$ is an injective VVS, but is not free. In this example, the set of prime elements is not empty, but it is not a generating set.

For example, if $n = 3, m = 2$, then $x_i = \rho^i(aaa), y_i = \rho^i(aaaa), i = 1,2$, are four different elements in Q, and: $g_i(x_1x_2y_1y_2) = \rho^i(aaaaaaa) = g_i(y_1y_2x_1x_2)$, which implies that $\mathbf{Q} = (Q,g)$ is not a free VVS.

In the above example the non free injective VVS is a vector valued subsemigroup of a free VVS, which is injective. The following example shows that it is possible to have a vector valued subsemigroup of a free (and so of an injective) VVS, which is not injective.

Example 4.2. Let $\mathbf{F} = (F(a), f)$ be the free (n,m)-semigroup generated by one element a, and let $m \geq 2$. For the elements of $F(a)$ we have defined their norm. In this example, $|a| = 1$, and $|\rho^i(x_1^t)| = \sum_{i=1}^t |x_i|$. Let $Q \subseteq F(a)$ be the set of all the elements whose norm is ≥ 4. Thus, $a \notin Q, \rho^i(aaa) \notin Q$. Let g, as an (n,m)-operation on Q, be the restriction of the (n,m)-operation f on $F(a)$. It follows directly from the definition that $\mathbf{Q} = (Q,g)$ satisfies *(Inj.1.)* We will show that it does not satisfy the condition *(Inj.2.)* Let $\mathbf{x} = \rho^1(\rho^1(aaa)\rho^2(aaa)a)\rho^2(aaaa)a$, and let $\mathbf{y} = \rho^1(aaaa)\rho^2(\rho^1(aaa)\rho^2(aaa)a)a$. Then $\mathbf{x} \neq \mathbf{y}$, and they are irreducible in $\mathbf{Q} = (Q,g)$, although they are reducible in $\mathbf{F} = (F(a), f)$. The definition of g implies that $g_i(\mathbf{x}) = f_i(\mathbf{x}) = \rho^i(aaaaa) = f_i(\mathbf{y}) = g_i(\mathbf{y}), i = 1,2$, i.e. $g(\mathbf{x}) = g(\mathbf{y})$. Hence, g is not an injection on the irreducible elements.

The next two examples are similar. The VVS in both of these examples are injective VVS, but they are not vector valued subsemigroups of a free VVS. In the first one, the set of prime elements is empty, while in the second one the set of prime elements is not empty, but it is not a generating set.

Example 4.3. Let $\mathbf{Q} = (Q, f)$ be a (3,2)-semigroup with the presentation

$$< a, b | f(aaa) = ba >,$$

i.e. a (3,2)-semigroup generated by two elements a and b with the relation $f(aaa) = ba$. We will give a short description of $\mathbf{Q} = (Q, f)$. Let $A_0 = \{a, b\}$, and let A_p be defined. We say that an $\mathbf{x} \in A_p^n, n \geq 3$, is not "good" if it has one of the following forms;

$$\mathbf{x}'aaa\mathbf{x}''; \qquad \mathbf{x}'\rho^1(\mathbf{u})\rho^2(\mathbf{u})\mathbf{x}''; \qquad \mathbf{x}'ab^k a\mathbf{x}'', k \geq 1,$$

where in $b^k = bb\dots b$, the element b appears k times. Otherwise, we say that $\mathbf{x}$ is "good". Next we define:

$$A_{p+1} = A_p \cup \{\rho^i(\mathbf{x}) | \mathbf{x} \in A_p^n, n \geq 3, i = 1, 2, \text{ and } \mathbf{x} \text{ is "good"}\}.$$

Let $Q = \cup_{s=0}^{\infty} A_s$. We have the same notion of "good" elements in $Q^{2,1}$. We define the norm $|x_1^n|$ of an $x_1^n \in Q^n$, for $n \geq 1$ as follows:

$$|a| = |b| = 1; \quad |x_1^n| = \sum_{i=1}^{n} |x_i|; \quad |\rho^j(\mathbf{x})| = |\mathbf{x}|.$$

We also define a **size** $[x_1^n]$ of an $x_1^n \in Q^n$, for $n \geq 1$ by:

$$[a] = 1; \quad [b] = 2; \quad [\rho^j(\mathbf{x})] = [\mathbf{x}]$$

and

$$[x_1^n] = \sum_{i=1}^{n} (|x_1^{i-1}| + 1) \cdot [x_i],$$

where by definition $|x_1^0| = 0$. Then, it can be checked by induction on the size, that $\mathbf{Q} = (Q, f)$ is a (3,2)-semigroup, where $f : Q^{2,1} \to Q^2$ is defined by:

(d.1) $f(aaa) = ba$;

(d.2) $f(\mathbf{x}'\rho^1(\mathbf{u})\rho^2(\mathbf{u})\mathbf{x}'') = f(\mathbf{x}'\mathbf{u}\mathbf{x}'')$, if not (d.1);

(d.3.0) $f(\mathbf{x}'aaa\mathbf{x}'') = f(\mathbf{x}'ba\mathbf{x}'')$, if not (d.1) and (d.2);

....

(d.3.k) $f(\mathbf{x}'ab^k a\mathbf{x}'') = f(\mathbf{x}'b^k aa\mathbf{x}'')$, for $k \geq 1$, if not (d.1), (d.2) and (d.3.j) for $0 \leq j < k$;

(d.4) $f(\mathbf{x}) = \rho^1(\mathbf{x})\rho^2(\mathbf{x})$, otherwise, i.e. for $\mathbf{x}$ "good".

It follows directly from the definition that $\mathbf{Q} = (Q, f)$ satisfies *(Inj.1.)*, and that the set of prime elements in (Q, f) is empty. A sequence $\mathbf{x} \in Q^{2,1}$ is irreducible if it is not of the form $\mathbf{x}'f(\mathbf{u})\mathbf{x}''$, which implies that $\mathbf{x} \neq \mathbf{x}'ba\mathbf{x}''$, since $ba = f(aaa)$. So, the only difference between the irreducible and "good" elements in $Q^{2,1}$ is that irreducible elements can have the form $\mathbf{x}'aaa\mathbf{x}''$. If $\mathbf{x}, \mathbf{y}$ are irreducible and $f(\mathbf{x}) = f(\mathbf{y})$, then both of them are "good", or both of them are not "good". If $\mathbf{x}, \mathbf{y}$ are "good", then $f(\mathbf{x}) = f(\mathbf{y})$ implies $\mathbf{x}=\mathbf{y}$. If $\mathbf{x}, \mathbf{y}$ are not "good" then: $\mathbf{x} = \mathbf{u}_1 a^k \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 a^t \mathbf{v}_2$, for some $k, t \geq 3$, where $\mathbf{u}_i$ does not end on a, and $\mathbf{v}_i$ does not begin by a, for i=1,2, and these a^k and a^t are the first appearances of a^s in $\mathbf{x}$ and $\mathbf{y}$. Then, $f(\mathbf{x}) = f(\mathbf{u}_1 ba^{k-2}\mathbf{v}_1), f(\mathbf{y}) = f(\mathbf{u}_2 ba^{t-2}\mathbf{v}_2)$, and these will be the first appearances of ba in $f(\mathbf{x})$ and $f(\mathbf{y})$. All this implies that $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$, and moreover that k=t. After finitely many such conclusions it will follow that $\mathbf{x}=\mathbf{y}$. Hence, $\mathbf{Q} = (Q, f)$ is an injective (3,2)-semigroup. The fact that $f(aaa) = ba$, implies not only that $\mathbf{Q} = (Q, f)$ is not a free (3,2)-semigroup, but also that $\mathbf{Q} = (Q, f)$ is not a (3,2)-subsemigroup of a free (3,2)-semigroup.

Example 4.4. Let (Q, f) be a (3,2)-semigroup with the presentation

$$< a, b, c | f(aaa) = ba >,$$

i.e. a (3,2)-semigroup generated by three elements a, b and c with the relation $f(aaa)$=ba. A similar discussion as in Example 4.3, implies that (Q, f) is an injective (3,2)-semigroup, but it is not a free (3,2)-semigroup and is not a (3,2)-subsemigroup of a free (3,2)-semigroup. The set P of prime elements in (Q, f) is not empty, but (Q, f) is not generated by P.

5. Commutative and fully commutative VVS

For a set Q, let $Q^{(t)}$ be its symmetric cartesian product, for $t \geq 1$, and let $\pi : Q^t \to Q^{(t)}$ be the projection map. Thus, $\pi(\mathbf{x}) = \pi(\mathbf{y})$ if and only if $\mathbf{y}$ is a permutation of $\mathbf{x}$. When there is no confusion we will denote $\pi(\mathbf{x})$ by $\mathbf{x}$. Let $Q^{(+)}$ be the union of all the symmetric cartesian products $Q^{(t)}$, for $t \in \mathrm{N}$. We denote by the same letter $\pi : Q^+ \to Q^{(+)}$ the projection map. This definition, and the fact that Q^+ is the free semigroup with the basis Q, implies that $Q^{(+)}$ is the free commutative semigroup with the basis Q.

Let $n = m + k$, be as in part 1. An (n, m)-semigroup (Q, f) is called *commutative*, (denoted by CVVS) if for each $\mathbf{x}, \mathbf{y} \in Q^n, \pi(\mathbf{x}) = \pi(\mathbf{y})$ implies $f(\mathbf{x}) = f(\mathbf{y})$. If (Q, f) is a commutative (n, m)-semigroup, then it can be described as a pair (Q, f), where $f : Q^{(n)} \to Q^m$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{(n+k)}, \mathbf{y}, \mathbf{u} \in Q^{(n)}$. Using the same notion of length of $\mathbf{x}$ as in part 1, let $Q^{(m,k)} = \{\pi(\mathbf{x}) | \mathbf{x} \in Q^+, |\mathbf{x}| = m + sk\}$. A commutative (n, m)-semigroup can be thought of as a pair (Q, f), where $f : Q^{(m,k)} \to Q^m$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{(m,k)}$ and $\mathbf{y}, \mathbf{u} \in Q^{(m,k)}$. All the notions from VVS are defined in the same way for CVVS, replacing Q^n and $Q^{m,k}$ by $Q^{(n)}$ and $Q^{(m,k)}$. The description of free CVVS is the same as the description of free VVS, using free commutative semigroups and symmetric cartesian product, i.e. using the free commutative algebra $\mathbf{F} = (F; \rho^1, \rho^2, \ldots, \rho^m)$ with the basis B, of type $\Omega = \{\rho_r^j | j \in \mathbf{N}_m, r \geq 1\}, F^{(+)}, F^{(m,k)}, F^{(m+rk)}, F(B)^{(+)}, F(B)^{(m,k)}$ and $F(B)^{(m+rk)}$, instead of their noncommutative analogs. The definition of injective CVVS is the same as the definition of injective VVS, and the same theorem holds.

Theorem 5.1. *A CVVS is free if and only if it is injective and is generated by the set of its prime elements.*

A fully commutative (n, m)-semigroup (denoted by FCVVS) is a pair $\mathbf{Q} = (Q, f)$ where $f : Q^{(n)} \to Q^{(m)}$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{(n+k)}, \mathbf{y}, \mathbf{u} \in Q^{(n)}$. A fully commutative (n, m)-semigroup can be thought of as a pair (Q, f), where $f : Q^{(m,k)} \to Q^{(m)}$, such that $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$ for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{(m,k)}$ and $\mathbf{y}, \mathbf{u} \in Q^{(m,k)}$.

The main difference of VVS and CVVS from FCVVS is the fact that in the last case there are no component operations, i.e. the fully commutative (n, m)-operation $f : Q^{(n)} \to Q^{(m)}$ can not be considered as m n-ary operations. On the other hand, there is a natural functor H from CVVS to FCVVS, and there are infinitely many functors K from FCVVS to CVVS, such that $H \circ K$ is the identity. The functor H is defined by:

If $\mathbf{Q} = (Q, f)$ is a commutative (n, m)-semigroup, then $H(\mathbf{Q}) = (Q, g)$ is the fully commutative (n, m)-semigroup defined by $g = \pi \circ f$, where $\pi : Q^m \to Q^{(m)}$ is the projection map.

For any map $\xi : Q^{(m)} \to Q^m$ such that $\pi \circ \xi = id$, and a fully commutative (n, m)-semigroup $\mathbf{Q} = (Q, f)$, $K(\mathbf{Q}) = (Q, g)$ is a commutative (n, m)-semigroup, defined by $g = \xi \circ f$.

The definitions of H and K imply that $H \circ K = id$.

The notions of generating sets, subsemigroups, and homomorphisms are defined as usual in the category of FCVVS. The free FCVVS with the basis B is a FCVVS $\mathbf{F}$ generated by B such that any map from B to a FCVVS $\mathbf{Q}$ extends to a FCVVS homomorphism form $\mathbf{F}$ to $\mathbf{Q}$. We note here that the previously mentioned extension in almost all the cases is not unique. Description of free FCVVS, using the description of free CVVS and the above mentioned functors is as follows.

Proposition 5.2. *If* $\mathbf{F} = (F(B), f)$ *is the free commutative* (n, m)*-semigroup with the basis* B*, then the fully commutative* (n, m)*-semigroup* $H(\mathbf{F}) = (F(B), \pi \circ f)$*, is the free fully commutative* (n, m)*-semigroup with the basis* B*.*

Since for FCVVS we do not have the component operations, we will state the notion for injective FCVVS in its language.

A FCVVS $\mathbf{Q} = (Q, f)$ is called *injective*, if:

(Finj.1.) For each $\mathbf{x}, \mathbf{y} \in Q^{(m)}$ in the image of f, $cn(\mathbf{x})$ consists of exactly m elements, and if $cn(\mathbf{x}) \cap cn(\mathbf{y}) \neq \emptyset$, then $cn(\mathbf{x}) = cn(\mathbf{y})$; and

(Finj.2.) If $\mathbf{x}, \mathbf{y} \in R(\mathbf{Q})$ and $f(\mathbf{x}) = f(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$, i.e. the restriction of f on $R(\mathbf{Q})$ is an injection.

It is easy to check the following:

Proposition 5.3. *Let* $\mathbf{Q} = (Q, f)$ *be a CVVS and let* $H(\mathbf{Q}) = (Q, g)$ *be its image under the functor* H*. Then:*

(a) If (Q, f) *is injective CVVS, then* (Q, g) *is injective FCVVS;*

(b) If P *is the set of primes in* (Q, f)*, then* P *is the set of primes in* (Q, g)*.*

The next proposition shows the converse.

Proposition 5.4. *Let* $\mathbf{Q} = (Q, f)$ *be a FCVVS, and let* $\xi : Q^{(m)} \to Q^m$ *be a map, such that* $\pi \circ \xi = id$*. Let* $K(\mathbf{Q}) = (Q, g)$ *be its image under the functor* K *for the map* ξ*. Then:*

(a) If (Q, g) *is generated by* B*, then* (Q, g) *is generated by* B*;*

(b) If P *is the set of prime elements in* (Q, f)*, then* P *is the set of prime elements in* (Q, g)*;*

(c) If (Q,f) is injective FCVVS, then (Q,g) is injective CVVS.

Proof. (a) If $a \in Q$ is such that $a \in cn(f(\mathbf{x}))$ for some $\mathbf{x} \in B^{(m,k)}$, then it follows that $a \in cn(g(\mathbf{x}))$. Further, an easy induction implies that B is a generating set for (Q,g).

(b) If $a \in Q$ is not prime in (Q,f), then $a \in cn(f(\mathbf{x}))$ for some $\mathbf{x} \in Q^{(m,k)}$, which implies that $a \in cn(g(\mathbf{x}))$, i.e. a is not prime in (Q,g). Conversely, if a is not prime in (Q,g), then $a \in cn(g(\mathbf{x}))$ for some $\mathbf{x} \in Q^{(m,k)}$, which implies that $a \in cn(f(\mathbf{x}))$, i.e. a is not prime in (Q,f).

(c) Let $\mathrm{im}\ g_j \cap \mathrm{im}\ g_i \neq \emptyset$. Then, there are $\mathbf{x},\mathbf{y} \in Q^{(m,k)}$, such that $g_j(\mathbf{x}) = g_i(\mathbf{y})$. Let $\mathbf{u} = g(\mathbf{x})$ and $\mathbf{v} = g(\mathbf{y})$. Then $g_j(\mathbf{x}) = g_i(\mathbf{y})$ together with $\pi(\mathbf{u}) = \pi \circ \xi \circ f(\mathbf{x}) = f(\mathbf{x}) = \mathbf{a}$ and $\pi(\mathbf{v}) = \pi \circ \xi \circ f(\mathbf{y}) = f(\mathbf{y}) = \mathbf{b}$ and *(Finj.1.)* implies that $cn(\mathbf{a}) = cn(\mathbf{b})$. Further, since $cn(\mathbf{a})$ and $cn(\mathbf{b})$ have exactly m elements, by *(Finj.1.)*, it follows that $\mathbf{a} = \mathbf{b} = f(\mathbf{x}) = f(\mathbf{y}) = \pi(\mathbf{u}) = \pi(\mathbf{v})$ has exactly m elements. Also, $\mathbf{u} = g(\mathbf{x}) = \xi \circ f(\mathbf{x}) = \xi(\mathbf{a}) = \xi(\mathbf{b}) = \xi \circ f(\mathbf{y}) = g(\mathbf{y}) = \mathbf{v}$. If $i \neq j$, then $cn(\pi(\mathbf{u})) = cn(\pi(\mathbf{v}))$ would have less than m elements. Hence, $i=j$. Next, for $g_j(\mathbf{x}) = g_j(\mathbf{y})$, the same discussion as above shows that $g(\mathbf{x}) = g(\mathbf{y})$, i.e. (Q,g) satisfies Inj. 1.

If $\mathbf{x},\mathbf{y} \in Q^{(m,k)}$ are irreducible for (Q,g), then they are also irreducible for (Q,f). So, if $g(\mathbf{x}) = g(\mathbf{y})$, then $\xi \circ f(\mathbf{x}) = \xi \circ f(\mathbf{y})$, and after composing with π, it follows that $f(\mathbf{x}) = \pi \circ \xi \circ f(\mathbf{x}) = \pi \circ \xi \circ f(\mathbf{y}) = f(\mathbf{y})$. Since (Q,f) is injective, it follows that $\mathbf{x}=\mathbf{y}$, implying that (Q,g) is injective CVVS. □

Now, Propositions 5.2, 5.3 and 5.4 imply the following

Theorem 5.5. *A FCVVS is free if and only if it is injective and is generated by the set of prime elements.*

References

[1] Čupona, Ǵ., Vector valued semigroups, Semigroup Forum 26 (1983), 65–74.

[2] Čupona, Ǵ., Celakoski, N., Markovski, S. and Dimovski, D., Vector valued groupoids, semigroups and groups, in: "Vector Valued Semigroups and Groups", pp. 1–79, Maced. Acad. Sci. and Arts, 1988.

[3] Čupona, Ǵ., Markovski, S., Dimovski, D. and Janeva, B., A combinatorial theory of vector valued semigroups; in: "Vector Valued Semigroups and Groups", pp. 141–183, Maced. Acad. Sci. and Arts, 1988.

[4] Dimovski, D., Free vector valued semigroups, in: Proc. 5th Conf. "Algebra & Logic", (Cetinje, 1986), pp. 55–62, University of Novi Sad, 1988.

[5] Trpenovski, B. and Čupona, Ǵ., $[m,n]$-groupoids, Bulletin Maced. Math. Phys. Soc. 21 (1970), 19–29.

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