RECTANGULAR n-BANDS

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A b s t r a c t: In this article we give a structure description of the class of rectangular n-bands which we introduce below. In doing this we generalize one of the results of [3], taking $n > 2$ to be arbitrary.

Let (S, f) be a n-groupoid, $n > 2$, i.e. (S, f) is a pair of a nonempty set S and a $(n + 1)$-ary operation defined in S. here we take $n > 2$ because for $n = 1$, as we can see later on, the problem reduces to the anticommutative semigroups of idempotents (see, for example [2]), and for $n = 2$ we reffer to [3].

In what follows we shall make use of the following short notations : instead of

$$f(x_0, x_1, \dots, x_n),\ \text{or}\ f(\dots, x_i, x_{i+1}, \dots, x_j, \dots),\ i < j,\ \text{or}\ f(\dots, x, x, \dots, x, \dots),$$

x apearing k times, we shall write $f(x_0^n)$, or $f(\dots, x_i^j, \dots)$, or $f(\dots, \underline{x}^k, \dots)$, respectively. Further , we shall simply write S instead (S, f).

We call S an anticyclic n-gropoid iff the following holds

(AC) $$f(x_0^n) = f(x_1^n, x_0) = \dots = f(x_n, x_0^{n-1}) \Rightarrow x_0 = x_1 = \dots = x_n.$$

Let μ_i, $i = 1, 2, \dots, n-1$ be equivalence relations on S. Then S is said to be a weak - associative n-groupoid with respect to $(\mu_1, \mu_2, \dots, \mu_{n-1})$ iff the following equalities hold ;

(WA) $$f\left(x_0^{i-1}, f\left(x_i^{i+n}\right), x_{i+n+1}^{2n}\right) = f\left(x_0^{i+k-1}, f\left(x_{i+k}^{i+k+n}\right), x_{i+k+n+1}^{2n}\right) \Leftrightarrow x_i \mu_i x_{2i}\ \text{for}\ i \neq \quad ;$$

$$x_{i+n+k}\, \mu_{i+k}\, x_{2(i+k)},\ \text{for}\ k \neq n - i;$$

$$x_j \mu_j x_{n+j},\ \text{for}\ j = i+1, i+2, \dots, i+k-1,$$

where $i = 0, 1, \dots, n-1$ and $k = 1, 2, \dots, n-i$.

Lemma 1. Every anticyclic weak-associative n-groupoid S is idempotent.

Proof. For any $x \in S$ we have that

$$f\left(f\left(\underline{x}^{n+1}\right),\underline{x}^{n}\right)=f\left(x,f\left(\underline{x}^{n+1}\right),\underline{x}^{n-1}\right)=\cdots=f\left(\underline{x}^{n},f\left(\underline{x}^{n+1}\right)\right)$$

which implies that $f\left(\underline{x}^{n+1}\right)=x$. □

An anticyclic weak-associative n-groupoid S is said to be a rectangular n-band iff the equivalences μ_i $i=1,2,\ldots,n-1$ are defined in the following way:

$$x\mu_i y \Leftrightarrow \left(\forall a_j \in S\right) f\left(a_1^i,x,a_{i+1}^n\right)=f\left(a_1^i,y,a_{i+1}^n\right),$$
$$i=1,2,\ldots,n-1,\ \ j=1,2,\ldots,n$$

Example. Let $A_j, j=0,1,\ldots,$ be nonempty sets and $S=A_0\times A_1\times\cdots\times A_n$. If $x_i=\left(a_{i0},a_{i1},\ldots,a_{in}\right)$ where $a_{ij}\in A_j$, $i,j=0,1,\ldots,n$ and if we put $f\left(x_0^n\right)=\left(a_{00},a_{11},\ldots,a_{nn}\right)$ then (S,f) will be a rectangular n-band. Let us denote this n-band by $[S,f]_d$. Observe that, if $x=\left(x_1,x_2,\ldots,x_n\right)$, $y=\left(y_0,y_1,\ldots,y_n\right)$ then $x\mu_i y$ iff $x_i=y_i$ for $i=1,2,\ldots,n-1$.

In what follows, S will stand for a rectangular n-band.

Lemma 2. Let $a,x_i\in S$, $i=1,2,\ldots,n$.

(i) If $x_i\mu_i a$, $i=1,2,\ldots,n-1$ then $f\left(a,x_1^{n-1},a\right)=a$;

(ii) if $f\left(a,x_1^n\right)=a$, then $a\mu_i x_i$ and if $f\left(x_1^{n-1},a\right)=a$ then $a\mu_i x_{i+1}$ for $i=1,2,\ldots,n-1$.

Proof. For the first part, from the definitions of μ_i and Lemma 1. it follows that

$$f\left(a,x_1^{n-1},a\right)=f\left(a,\underline{a}^{n-1},a\right)=f\left(\underline{a}^{n+1}\right)=a.$$

Now , let $f\left(a,x_1^n\right)=a$; from

$$f\left(f\left(\underline{a}^{n+1}\right),x_1^n\right)=a=f\left(\underline{a}^{n},f\left(a,x_1^n\right)\right),$$

by (WA) it follows that $a\mu_i x_i$, $i=1,2,\ldots,n-1$. Similarly, for the second part of (ii), we can get that $a\mu_i x_{i+1}$, $i=1,2,\ldots,n-1$. □

Lemma 3. For arbitrary $x,y\in S$ the following are true :

(i) $f\left(\underline{x}^i,y,\underline{x}^{n-i}\right)\mu_j x$ for $i=0,1,\ldots,n; j=1,2,\ldots n-1; i\neq j$;

(ii) $f(\underline{x}^i, y, \underline{x}^{n-i})\mu_j y$ for $i = 1,2,\ldots,n-1$.

Proof. Let $z = f(\underline{x}^n, y)$. By Lemma 2(ii) from

$$f(\underline{x}^n, z) = f(\underline{x}^n, f(\underline{x}^n, y)) = f(f(\underline{x}^{n+1}), \underline{x}^{n-1}, y) = f(\underline{x}^n, y) = z$$

it follows that $z\mu_i x$ for $i = 1,2,\ldots,n-1$. So, taking into account the symetry as well, we have that

(1) $$f(\underline{x}^n, y)\mu_i x\mu_i f(y, \underline{x}^n) \text{ , i=1,2,...,n-1.}$$

Let $i \neq n-1$, $u = f(\underline{x}^{n-1}, y, x)$. Again by Lemma 2(ii), the symetry, and (1), from $f(\underline{x}^i, u, \underline{x}^{n-i}) = f(\underline{x}^i, f(\underline{x}^{n-1}, y, x), \underline{x}^{n-i}) =$
$= f(\underline{x}^{i-1}, f(\underline{x}^n, y), \underline{x}^{n-i+1}) = f(\underline{x}^{i-1}, x, \underline{x}^{n-i+1}) = x$, it follows that

(2)
$$f(\underline{x}^{n-1}, y, x)\mu_i x \quad for\ i = 1,2,\ldots,n-2;$$
$$f(x, y, \underline{x}^{n-1})\mu_i x \quad for\ i = 2,3,\ldots,n-1.$$

Now, let us put $v = f(\underline{x}^i, y, \underline{x}^{n-i})$, $i = 1,2,\ldots,n-1$. By (2), since $n > 2$, we have that $f(v, \underline{x}^{i-1}, y, \underline{x}^{n-i}) = f(f(\underline{x}^i, y, \underline{x}^{n-i}), \underline{x}^{i-1}, y, \underline{x}^{n-i}) =$
$= f(\underline{x}^i, y, f(\underline{x}^{n-1}, y, x), \underline{x}^{n-i-1}) = f(\underline{x}^i, y, x, \underline{x}^{n-i}) = v$, which, by Lemma 2(ii) implies that $v\mu_i y$ and $v\mu_j x$ $\quad for\ j = 1,2,\ldots,n-1, \quad j \neq i$. □

Lemma 4. Let $x_i \in S$, $i = 0,1,\ldots,n$.

(i) If $x_0\mu_i x_i$ and $x_i\mu_{i+1}x_{i+2}$, $i = 1,2,\ldots,n-2$ then $f(x_0^n)\mu_i x_i$ if $x_0\mu_{n-1}x_{n-1}$ then $f(x_0^n)\mu_{n-1}x_{n-1}$;

(ii) If $x_n\mu_i x_i$ and $x_i\mu_{i-1}x_{i-2}$, $i = 2,3,\ldots,n-1$ then $f(x_0^n)\mu_i x_i$ if $x_n\mu_1 x_1$ then $f(x_0^n)\mu_1 x_1$.

Proof. For the resons of symetry, we shall prove only part (i). Let $i \neq n-1$. Since $x_0\mu_i x_i$ and $x_i\mu_{i+1}x_{i+2}$ by Lemma 2 we have that

$$w = f(\underline{x}_i^i, f(x_0^n), \underline{x}_i^{n-i}) = f(\underline{x}_i^i, x_0, f(x_1^n, x_i), \underline{x}_i^{n-i-1}) =$$

$$= f\left(\underline{x}_i^{i+1}, f\left(x_1^n, x_i\right), \underline{x}_i^{n-i-1}\right)$$

implies that $w\mu_i f\left(x_0^n\right)$ and $w\mu_i x_i$, so that $f\left(x_0^n\right)\mu_i x_i$.

If $i = n-1$ and $x_0 \mu_{n-1} x_{n-1}$ then

$$w = f\left(\underline{x}_{n-1}^{n-1}, f\left(x_0^n\right), x_{n-1}\right) = f\left(\underline{x}_{n-1}^{n-1}, x_0, f\left(x_1^n, x_{n-1}\right)\right) = f\left(\underline{x}_{n-1}^n, f\left(x_1^n, x_{n-1}\right)\right)$$

implies that $w\mu_{n-1}x_{n-1}$ and $w\mu_{n-1} f\left(x_0^n\right)$ and so, $f\left(x_0^n\right)\mu_{n-1}x_{n-1}$. □

Lemma 5. For every $x_j \in S$, $j = 0,1,\ldots,n$ and for every $i = 1,2,\ldots,n-1$ $f\left(x_0^n\right)\mu_i x_i$.

Proof. First we shall proof that there exists $y \in S$ such that $y\mu_1 x_1$ and $f\left(x_0^n\right) = f\left(x_0, x_1, y, x_3^n\right)$. By Lemma 3(i), since n > 2, we have that $f\left(\underline{x}_1^3, x_2, \underline{x}_1^{n-3}\right)\mu_1 x_1$. Let us put $y = f\left(\underline{x}_1^2, x_2, \underline{x}_1^{n-3}, x_2\right)$; according to Lemma 4, we have that $y\mu_1 x_1$. Now,

$$f\left(x_0^n\right) = f\left(x_0, f\left(\underline{x}_1^3, x_2, \underline{x}_1^{n-3}\right), x_2^n\right) = f\left(x_0, x_1, f\left(\underline{x}_1^2, x_2, \underline{x}_1^{n-3}, x_2\right), x_3^n\right) =$$
$$= f\left(x_0, x_1, y, x_3^n\right).$$

Let us put $u = f\left(x_0^n\right)$; since $y\mu_1 x_1$ and by Lemma 4 (ii) $f\left(\underline{x}_1^2, x_3^n, x_1\right)\mu_1 x_1$, we have that

$$f\left(u, x_1^n\right) = f\left(f\left(x_0^n\right), x_1^n\right) = f\left(f\left(x_0, x_1, y, x_3^n\right), x_1^n\right) = f\left(x_0, f\left(x_1, y, x_3^n, x_1\right), x_2^n\right) =$$
$$= f\left(x_0, f\left(\underline{x}_1^2, x_3^n, x_1\right), x_2^n\right) = f\left(x_0, x_1, x_2^n\right) = f\left(x_0^n\right) = u,$$

which, by Lemma 2(ii), implies that $u\mu_i x_i$ for $i = 1,2,\ldots,n-1$. □

Theorem 1. For every $i = 1,2,\ldots,n-1$ each equivalence class modulo μ_i is a rectangular n-subband of S. Further, every two rectangular n-subbands of S for the same i are isomorphic.

Proof. Let H be an equivalence class of S modulo μ_i. If $x_k \in H$, $k = 0,1,\ldots, n$, then from Lemma 5 it follows that $f\left(x_0^n\right) \in H$ which proves that H is an n-subgroupoid of S. It is obvious that (AC) and (WA) hold in H, which proves the first part of the Theorem.

Let H_1 and H_2 be two equivalence classes of S modulo μ_i for an arbitrary i = 1, 2,...,n-1 with $a \in H_1$ and $b \in H_2$. By putting

$$\varphi_{ab}(x) = f\left(\underline{x}^i, b, \underline{x}^{n-i}\right), \quad x \in H_1,$$

according to Lemma 5 we define a mapping from H_1 into H_2.

Let $\varphi_{ab}(x) = \varphi_{ab}(y)$, $x, y \in H_1$, *i.e.* $f\left(\underline{x}^i, b, \underline{x}^{n-i}\right) = f\left(\underline{y}^i, b, \underline{y}^{n-i}\right) = c \in H_2$.

Then $f\left(c, \underline{x}^{n-1}, c\right) = f\left(f\left(\underline{x}^i, b, \underline{x}^{n-i}\right), \underline{x}^{n-1}, c\right) = f\left(\underline{x}^i, f\left(b, \underline{x}^n\right), \underline{x}^{n-i-1}, c\right) =$

$= f\left(\underline{x}^n, c\right) = f\left(\underline{x}^n, f\left(\underline{x}^i, b, \underline{x}^{n-i}\right)\right) = f\left(\underline{x}^i, f\left(\underline{x}^n, b\right), \underline{x}^{n-i}\right) = f\left(\underline{x}^{n+1}\right) = x$.

With the above we have proved that

(3) $$f\left(c, \underline{x}^{n-1}, c\right) = x, \quad f\left(c, \underline{y}^{n-1}, c\right) = y.$$

From $f\left(\underline{x}^i, b, \underline{x}^{n-i}\right) = c = f\left(\underline{y}^i, b, \underline{y}^{n-i}\right)$ and $b\mu_i c$ it follows that

(4) $$f\left(\underline{x}^i, c, \underline{x}^{n-i}\right) = c = f\left(\underline{y}^i, c, \underline{y}^{n-i}\right),$$

which implies that $c\mu_j x, y$ for j =1,2,...,n-1, $j \neq i$. So,

(5) $$f\left(x, \underline{c}^{n-1}, x\right) = c = f\left(y, \underline{c}^{n-1}, y\right).$$

Now, since $x\mu_i y$ and $x\mu_j c\mu_j y$ for $j \neq i$, from (3) it follows that

$$x = f\left(c, \underline{x}^{n-1}, c\right) = f\left(c, \underline{y}^{n-1}, c\right) = y,$$

which proves that φ_{ab} is an injection.

Let $y \in H_2$ and let us put $x = f\left(\underline{y}^i, a, \underline{y}^{n-i}\right) \in H_1$. By Lemma 2(i) we have that $x\mu_j y$ for j =1,2,...,n-1, $j \neq i$, and since $y\mu_i b$, we get

$$\varphi_{ab}(x) = f\left(\underline{x}^i, b, \underline{x}^{n-i}\right) = f\left(x, \underline{y}^{n-1}, x\right) = f\left(f\left(\underline{y}^i, a, \underline{y}^{n-i}\right), \underline{y}^{n-1}, x\right) =$$

$$= f\left(\underline{y}^i, f\left(a, \underline{y}^n\right), \underline{y}^{n-i-1}, x\right) = f\left(\underline{y}^n, x\right) = f\left(\underline{y}^n, f\left(\underline{y}^i, a, \underline{y}^{n-i}\right)\right) =$$

$$= f\left(\underline{y}^i, f\left(\underline{y}^n, a\right), \underline{y}^{n-i}\right) = f\left(\underline{y}^{n+1}\right) = y,$$

which proves that φ_{ab} is a surjection, as well.

To complete the proof of the Theorem it remains to prove the homomorphic property of φ_{ab}. Let $x_j \in H_1$, j = 0,1,...,n and $b \in H_2$. Then from

$$\varphi_{ab}(x_j) = f\left(\underline{x}_j^i, b, \underline{x}_j^{n-i}\right) = c_j \in H_2$$

it follows that $x_j \mu_k c_j$, $k = 1,2,\ldots,n-1$, $k \neq i$ and $b\mu_i c_j$, which implies that

(6) $$\varphi_{ab}(x_j) = f\left(x_j, \underline{c}_j^{n-1}, x_j\right) = c_j,$$

from where, as in the proof of (3) it follows that

(7) $$f\left(c_j, \underline{x}_j^{n-1}, c_j\right) = x_j.$$

Now, $f(\varphi_{ab}(x_0), \varphi_{ab}(x_1), \ldots, \varphi_{ab}(x_n)) = f(c_0^n)$.

If we put $f(c_0^n) = c$ we will have that

(8) $$f(\varphi_{ab}(x_0), \varphi_{ab}(x_1), \ldots, \varphi_{ab}(x_n)) = f\left(c_0, \underline{c}^{n-1}, c_n\right) = c.$$

since $c\mu_j c_j$ for $j = 1,2,\ldots,n-1$.

On the other hand, since $f(x_0^n)\mu_j x_j \mu_j c_j \mu_j c$ for $j = 1,2,\ldots,n-1$, $j \neq i$ and $c\mu_i b$, we have that

$$\varphi_{ab}\left(f(x_0^n)\right) = f\left(\underline{f(x_0^n)}^i, b, \underline{f(x_0^n)}^{n-i}\right) = f\left(f(x_0^n), \underline{c}^{n-1}, f(x_0^n)\right)$$

and $f(x_0^n) = f\left(x_0, \underline{c}^{i-1}, x_i, \underline{c}^{n-i-1}, x_n\right)$. But $x_i \mu_i x_{i-1}$ so that

$$\varphi_{ab}\left(f(x_0^n)\right) = f\left(f\left(x_0, \underline{c}^{i-1}, x_{i-1}, \underline{c}^{n-i-1}, x_n\right), \underline{c}^{n-1}, f(x_0^n)\right) =$$
$$= f\left(x_0, f\left(\underline{c}^{i-1}, x_{i-1}, \underline{c}^{n-i-1}, x_n, c\right), \underline{c}^{n-2}, f(x_0^n)\right) = f\left(x_0, \underline{c}^{n-1}, f(x_0^n)\right).$$

since $x_{i-1}\mu_{i-1}c$ and taking $i > 1$. For $i = 1$ we come to the same result, as follows

$$\varphi_{ab}\left(f(x_0^n)\right) = f\left(f\left(x_0, x_0, \underline{c}^{n-2}, x_n\right), \underline{c}^{n-1}, f(x_0^n)\right) =$$
$$= f\left(x_0, f\left(x_0, \underline{c}^{n-2}, x_n, c\right), \underline{c}^{n-2}, f(x_0^n)\right) = f\left(x_0, \underline{c}^{n-1}, f(x_0^n)\right),$$

since $f\left(x_0, \underline{c}^{n-2}, x_n, c\right)\mu_1 c$. Further, since $x_i \mu_i x_{i+1}$, for $i < n-1$ we will have that

$$\varphi_{ab}\left(f(x_0^n)\right) = f\left(x_0, \underline{c}^{n-1}, f\left(x_0, \underline{c}^{i-1}, x_{i+1}, \underline{c}^{n-i-1}, x_n\right)\right) =$$
$$= f\left(x_0, \underline{c}^{n-2}, f\left(c, x_0, \underline{c}^{i-1}, x_{i+1}, \underline{c}^{n-i-1}\right), x_n\right) = f\left(x_0, \underline{c}^{n-1}, x_n\right),$$

and if $i = n-1$

$$\varphi_{ab}\left(f\left(x_0^n\right)\right)=f\left(x_0,\underline{c}^{n-1},f\left(x_0,\underline{c}^{n-2},x_n,x_n\right)\right)=$$
$$=f\left(x_0,\underline{c}^{n-2},f\left(c,x_0,\underline{c}^{n-2},x_n\right),x_n\right)=f\left(x_0,\underline{c}^{n-1},x_n\right).$$

For $j=0$ in (7) and if $i\neq 2$, then

$$\varphi_{ab}\left(f\left(x_0^n\right)\right)=f\left(f\left(\underline{c}_0^i,x_0,\underline{c}_0^{n-i}\right),\underline{c}^{n-1},x_n\right)=$$
$$=f\left(f\left(\underline{c}_0^i,x_0,\underline{c}_0^{n-i}\right),f\left(c_0,c,\underline{c}_0^{n-2},c\right),\underline{c}^{n-2},x_n\right)=$$
$$=f\left(f\left(f\left(\underline{c}_0^i,x_0,\underline{c}_0^{n-i}\right),c_0,c,\underline{c}_0^{n-2}\right),\underline{c}^{n-1},x_n\right)=$$
$$=f\left(f\left(c_0,f\left(\underline{c}_0^{i-1},x_0,\underline{c}_0^{n-i-1}\right),c,\underline{c}_0^{n-2}\right),\underline{c}^{n-1},x_n\right)=$$
$$=f\left(f\left(c_0,c_0,c,\underline{c}_0^{n-2}\right)\underline{c}^{n-1},x_n\right)=$$
$$=f\left(c_0,f\left(c_0,c,\underline{c}_0^{n-2},c\right),\underline{c}^{n-2},x_n\right)=f\left(c_0,\underline{c}^{n-1},x_n\right).$$

For $i=2$ we proceed as follows:

$$\varphi_{ab}\left(f\left(x_0^n\right)\right)=f\left(f\left(c_0,c_0,x_0,\underline{c}_0^{n-2}\right),f\left(c_0,c,\underline{c}_0^{n-2},c\right),\underline{c}^{n-2},x_n\right)=$$
$$=f\left(f\left(f\left(c_0,c_0,x_0,\underline{c}_0^{n-2}\right),c_0,c,\underline{c}_0^{n-2}\right),\underline{c}^{n-1},x_n\right)=$$
$$=f\left(f\left(c_0,c_0,f\left(x_0,\underline{c}_0^{n-1},c\right),\underline{c}_0^{n-2}\right),\underline{c}^{n-1},x_n\right)=$$
$$=f\left(f\left(\underline{c}_0^{n+1}\right),\underline{c}^{n-1},x_n\right)=f\left(c_0,\underline{c}^{n-1},x_n\right).$$

Repeating the above calculations in a symetrical way, with x_n instead of x_0, we finally come to $\varphi_{ab}\left(f\left(x_0^n\right)\right)=f\left(c_0,\underline{c}^{n-1},c_n\right)$ which together with (8) completes the proof of the Theorem. □

Let H be an equivalence class of 0 modulo μ_1. We define an n-ary operation g in H by

$$(*)\qquad g\left(x_1^n\right)=f\left(x_1,a,x_2^n\right),\quad a,x_i\in H.$$

Theorem 2. (H,g) is a rectangular $(n\text{-}1)$-band. If H_1, H_2 are any two equivalence classes of S modulo μ_1 and g_1, g_2 defined as $(*)$, then (H_1,g_1) and (H_2,g_2) are isomorphic.

Proof. For any $a,b \in H$ and every $x_i \in H$, $i = 1,2,\dots,n$ we have that $g(x_1^n) = f(x_1, a, x_2^n) = f(x_1, b, x_2^n)$ which means that g is well defined in H.

Let $g(x_1^n) = g(x_n, x_1^{n-1}) = g(x_{n-1}, x_n, x_1^{n-2}) = \dots = g(x_2^n, x_1)$, i.e.

$$f(x_1, a, x_2^n) = f(x_n, a, x_1^{n-1}) = f(x_{n-1}, a, x_n, x_1^{n-2}) = \dots = f(x_2, a, x_3^n, x_1) = b.$$

Now, since $a\mu_1 b$ and $x_i \mu_j b$ for $i = 1,2,\dots,n$; $j = 2,3,\dots,n-1$, we have that

$$f(x_1, \underline{b}^{n-1}, x_n) = f(x_n, \underline{b}^{n-1}, x_{n-1}) = \dots = f(x_{i+1}, \underline{b}^{n-1}, x_i) = \dots = f(x_2, \underline{b}^{n-1}, x_1) = b.$$

Let $i \neq 1, n$; from $f(x_{i+1}, \underline{b}^{n-1}, x_i) = f(x_i, \underline{b}^{n-1}, x_{i-1}) = b$ it follows that

$$x_i = f(\underline{x_i}^{n+1}) = f(x_i, \underline{b}^{n-1}, x_i) = f(x_i, \underline{b}^{n-2}, x_{i+1}, x_i) = f(x_i, \underline{b}^{n-2}, f(x_{i+1}, x_{i+1}, \underline{b}^{n-2}, x_{i+1}), x_i) =$$

$$= f(x_i, \underline{b}^{n-2}, x_{i+1}, f(x_{i+1}, \underline{b}^{n-2}, x_{i+1}, x_i)) = f(x_i, \underline{b}^{n-1}, f(x_{i+1}, \underline{b}^{n-1}, x_i)) =$$

$$= f(x_i, \underline{b}^{n-1}, f(x_i, \underline{b}^{n-1}, x_{i-1})) = f(f(x_i, \underline{b}^{n-1}, x_i), \underline{b}^{n-1}, x_{i-1}) = f(x_i, \underline{b}^{n-1}, x_{i-1}) = b.$$

In the same way, from $f(x_1, \underline{b}^{n-1}, x_n) = f(x_2, \underline{b}^{n-1}, x_1) = b$ it follows that $x_1 = b$ and similarly, from $f(x_1, \underline{b}^{n-1}, x_n) = f(x_n, \underline{b}^{n-1}, x_{n-1}) = b$ it follows that $x_n = b$. Thus $x_1 = x_2 = \dots = x_n$ which means that (H,g) is anticyclic. It is almost obvious that (H, g) is a weak-associative $(n - 1)$ - groupoid with respect to $(\mu_1, \mu_2, \dots, \mu_{n-1})$, which proves the first part of the Theorem.

The second part of this Theorem follows from the second part of Theorem 1. Namely if

$$\varphi_{ab}(g_1(x_1^n)) = \varphi_{ab}(f(x_1, a, x_2^n)) = f(\varphi_{ab}(x_1), \varphi_{ab}(a), \varphi_{ab}(x_2), \dots, \varphi_{ab}(x_n)) =$$

$$g_2(\varphi_{ab}(x_1), \varphi_{ab}(x_2), \dots, \varphi_{ab}(x_n)). \square$$

Let S be an n-semigroup. We call S a <u>k-zero n-semigroup</u> iff for every $x_i \in S$, $i = 0,1,\dots,n$ $f(x_0^n) = x_k$, $k = 0,1,\dots,n$.

Now we are ready to give a structure description of a rectangular n-band

Theorem 3. An n-groupoid S is a rectangular n-band iff S is isomorphic to a direct product $A_0 \times A_1 \times \dots \times A_n$ where A_k is a k-zero n-semigroup, $k = 0,1,\dots,n$.

Proof. According to [1], [2] and [3] the Theorem is true for $n = 1$ and $n = 2$. Further we proceed by induction. Let H be an arbitrary, but fixed equivalence class of S modulo μ_1. Let $A_0, A_2, A_3, \ldots, A_n$ be nonempty sets, such that H is isomorphic to $A_0 \times A_2 \times A_3 \times \cdots \times A_n$ with A_0 0-zero, A_2 1-zero,..., A_n $(n-1)$ - zero $(n-1)$-semigroups. In what follows, we take A_j , $j = 0, 2, 3, \ldots, n$ to be simply nonempty sets. So, there is a bijection $\psi: H \to A_0 \times A_2 \times A_3 \times \cdots \times A_n$.

Let us denote by A_1 the index set for the family $\{H_j | j \in A_1\}$ of all equivalence classes of S modulo μ_1. Then $S = \bigcup \{H_j | j \in A_1\}$ where the union is disjoint. Let $\varphi_j : H \to H_j$, $j \in A_1$ (if $H = H_j$ then φ_j is the identity on H) be the isomorphism defined in Theorem 1. Let us put

$$\overline{S} = A_0 \times A_1 \times \cdots A_n$$

and in $\overline{S}$ we define an $(n+1)$-ary operation as in the Example, i.e. we take A_j to be a j-zero n-semigroup for $j = 0, 1, 2, \ldots, n$. Now , let $\chi : S \to \overline{S}$ be defined as follows : if $x \in H_{a_1}$, $a_1 \in A_1$ and if $\psi\varphi_{a_1}^{-1}(x) = (a_0, a_2, \ldots, a_n)$ we put $\chi(x) = (a_0, a_1, a_2, \ldots, a_n)$. It is easely seen that χ is a bijection and obviously, taking into consideration that A_j is a j-zero n-semigroup for $j = 0, 1, 2, \ldots, n$, it turns out that χ is an isomorphism. This proves the Theorem in one direction. The proof in the oposite direction is established by the Example. □

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