

FREE GROUPOIDS WITH $xy^2 = xy$

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Abstract

The main results of the paper are Theorems 1, 2, 3. Theorem 1 gives a canonical description of free objects in the variety \mathcal{U}_r of groupoids which satisfy the identity $xy^2 = xy$. In Theorem 2 the class of \mathcal{U}_r -free groupoids is characterized within the class of \mathcal{U}_r -injective groupoids, which is larger than the class of \mathcal{U}_r -free groupoids. Finally, in Theorem 3, it is shown that the class of \mathcal{U}_r -free groupoids is hereditary, and that a \mathcal{U}_r -free groupoid with rank 2 contains subgroupoids with infinite rank.

0. Introduction

Throughout the paper we denote by $F = (F, \cdot)$ a free groupoid (in the class of all groupoids) with a given basis B . It is well-known (for example [1; I.1]) that the following two properties characterize F :

- (a) F is injective, i.e. the mapping $\cdot : (a, b) \mapsto ab$ is an injection from F^2 into F .
- (b) The set B of primes in F generates F . (If $G = (G, \cdot)$ is a groupoid, and $a \in G \setminus GG$, then we say that a is a prime in G .)

As usual, if $G = (G, \cdot)$ is a groupoid, and n is a positive integer, then the transformation $x \mapsto x^n$ is defined as follows:

$$x^1 = x, \quad x^{k+1} = x^k x. \quad (0.1)$$

An element $a \in G$ is called a *proper power* in G iff there exist a $b \in G$ and $n \in \mathbb{N}$, $n \geq 2$ (\mathbb{N} is the set of positive integers), such that $a = b^n$. Then we say that b is a *base*, and n is an *exponent* of a in G .

It can be easily shown by (a) and (0.1) that, if u is a proper power in F , the base $t = \underline{u}$ and exponent $n = \text{ex}(u)$ are unique. If $u \in F$ is not a proper power in F , then we say that u is the *base* of u in F , and write $\underline{u} = u$; in this case, 1 is the *exponent* of u in F .

Notions as subgroupoids, homomorphisms, variety of groupoids, ... have usual meanings ([2]).

Now we can state the main results of the paper.

THEOREM 1. Let $R = (R, *)$ be defined as follows:

$$B \subset R \subset F \ \& \ (\forall u, v \in F) \{uv \in R \Leftrightarrow u, v \in R \ \& \ \underline{v} = v\}, \quad (0.2)$$

$$(\forall u, v \in R) u * v = u\underline{v}. \quad (0.3)$$

Then R is a free groupoid in \mathcal{U}_r with the (unique) basis B .

(We say that R is a *canonical \mathcal{U}_r -groupoid*.)

In order to state Th.2, we will define the notion of \mathcal{U}_r -injectivity. Namely, we say that a groupoid $H = (H, \cdot) \in \mathcal{U}_r$ is \mathcal{U}_r -*injective* iff it satisfies the following conditions:

$$1) (\forall a \in H, n \in \mathbb{N}) a \neq a^{n+1}.$$

2) For each $a \in HH$ there is a unique pair $(b, c) \in H^2$ such that $a = bc$ and:

$$2.1) (\forall d \in H, n \in \mathbb{N}) c \neq d^{n+1}.$$

$$2.2) (\forall b', c' \in H) [a = b'c' \Rightarrow b' = b \ \& \ (c' = c^m, \text{ for some } m \geq 1)].$$

In this case we say that b is the *left* and c is the *right divisor* of a (or shortly: (b, c) is the *pair of divisors* of a) and we write $b \mid a$, $c \mid a$. A sequence a_1, a_2, \dots of elements of H is called a *divisor chain* in H iff $a_{i+1} \mid a_i$ whenever a_{i+1} is a member of the sequence.

In Section 2 we give a complete description of the class of \mathcal{U}_r -injective groupoids, and show that it is larger than the class of \mathcal{U}_r -free ¹⁾ groupoids. The following property is a description of \mathcal{U}_r -free groupoids within the class of \mathcal{U}_r -injective groupoids.

THEOREM 2. If $H = (H, \cdot)$ is a \mathcal{U}_r -injective groupoid, then the following conditions are equivalent:

¹⁾ We will often say " \mathcal{U}_r -free groupoid" instead of "free groupoid in \mathcal{U}_r ".

- (i) \mathbf{H} is \mathcal{U}_r -free.
- (ii) There is a mapping $| \cdot | : a \mapsto |a|$ from \mathbf{H} into the set \mathbb{N} of positive integers such that: $b \mid a \Rightarrow |b| < |a|$.
- (iii) Every divisor chain in \mathbf{H} is finite.
- (iv) The set B of primes in \mathbf{H} generates \mathbf{H} .
Then B is the basis of \mathbf{H} .

THEOREM 3. (1) The class of \mathcal{U}_r -injective groupoids and the class of \mathcal{U}_r -free groupoids are hereditary.

(2) If \mathbf{H} is a \mathcal{U}_r -free groupoid with rank one, then each subgroupoid of \mathbf{H} is infinite and isomorphic to \mathbf{H} .

(3) If \mathbf{H} is a \mathcal{U}_r -free groupoid with rank two, then there exists subgroupoids of \mathbf{H} with infinite rank.

Theorem i ($i = 1, 2, 3$) (beside other auxiliary results) will be proved in Section i .

SOME REMARKS

1. The axiom $xy^2 = xy$ of \mathcal{U}_r suggests to consider the rewriting system (RS) on F induced by the elementary transformation $uv^2 \rightarrow uv$. Clearly, this system is terminating (T) but it is not Church-Rosser (CR) one (see [5; 2.9, 3.5]). For example, we have: $a \cdot a^2 a^2 \rightarrow a \cdot a^2 a$ and $a \cdot a^2 a^2 = a(a^2)^2 \rightarrow aa^2 \rightarrow a^2$. But, if we allow each transformation of the form $uv^k \rightarrow uv$, where $k \geq 2$, then we would obtain the corresponding RS which is a convenient TCR. We note that RS-s induced by $x^2 y^2 \rightarrow (xy)^2$ (i.e. $x^n \rightarrow x$) are convenient TCR for the variety \mathcal{V}_2 (\mathcal{V}) defined by $x^2 y^2 = (xy)^2$ ($x^n = x$, $n \geq 2$).

2. In [3], [4] corresponding Th. 1, Th. 2, Th. 3 for the varieties \mathcal{V}_2 and \mathcal{V} are shown. The formulation of these theorems for \mathcal{V}_2 ([3]) and \mathcal{V} ([4]) are almost the same as for \mathcal{U}_r , except Th. 3 for \mathcal{V}_2 (the class of \mathcal{V}_2 -free groupoids is not hereditary).

3. Denote by \mathcal{U}_l the variety of groupoids with the identity $x^2 y = xy$. Clearly:

$$\mathbf{G} = (G, \cdot) \in \mathcal{U}_l \Leftrightarrow \mathbf{G}^{\text{op}} = (G, \circ) \in \mathcal{U}_r,$$

where $x \circ y = yx$. Therefore, each \mathcal{U}_r -property can be translated into corresponding \mathcal{U}_l -property.

1. Canonical \mathcal{U}_r -groupoids

A proof of Th. 1 will be given below.

First, let $u \mapsto |u|$ be the homomorphism from F into the groupoid $(\mathbb{N}, +)$ which is an extension of $B \rightarrow \{1\}$. Then:

$$|b| = 1 \quad |uv| = |u| + |v|, \quad (1.1)$$

for any $b \in B$ and $u, v \in F$. (We say that $|u|$ is the *norm* of $u \in F$.) By induction on norm, the following relation can be easily shown:

$$(\forall u, v \in F, p, q \in \mathbb{N}) [u^{p+1} = v^{q+1} \Rightarrow u = v, p = q], \quad (1.2)$$

and this implies that $u \mapsto \underline{u}$, where \underline{u} is the base of u , is a well defined transformation of F , such that

$$(\forall v \in F) \underline{\underline{v}} = \underline{v}. \quad (1.3)$$

Moreover, (0.2), (0.3) and (1.1) imply:

$$v \in R \Rightarrow \underline{v} \in R, \quad (1.4)$$

$$n \geq 2 \Rightarrow (v^n \in R \Leftrightarrow v = \underline{v} \in R), \quad (1.5)$$

$$(\forall u, v \in R) |u| + 1 \leq |u| + |\underline{v}| \leq |u * v| \leq |u| + |v|, \quad (1.6)$$

$$|u * v| = |u| + |v| \Leftrightarrow v = \underline{v}.$$

As a corollary from (0.2), (0.3) and (1.4) we obtain:

1.1. $*$: $R^2 \rightarrow R$ is a well defined mapping, i.e. R is a groupoid. \square

Moreover, from (0.3) and (1.3) we obtain:

$$(\forall u, v \in R) u * (v * v) = u * (v \underline{v}) = u \underline{\underline{v}} = u \underline{v} = u * v.$$

Therefore:

1.2. $R \in \mathcal{U}_r$. \square

It is also clear that:

1.3. B is the least generating subset of R . \square

In completing the proof of Th. 1 we will use the next two properties of \mathcal{U}_r .

1.4. The following identities hold in \mathcal{U}_r :

$$xy^n = xy, \quad x^m x^n = x^{m+1}, \quad (x^m)^n = x^{m+n-1}, \quad \text{for any } n, m \in \mathbb{N}.$$

Proof. Assuming $xy^n = xy$, we obtain:

$$xy^{n+1} = x \cdot y^n y = x \cdot y^n y^n = x(y^n)^2 = xy^n = xy.$$

The other two identities are trivial corollaries of the first one. \square

1.5. If $\mathbf{G} = (G, \cdot) \in \mathcal{U}_r$, and φ is a homomorphism from \mathbf{F} into \mathbf{G} , then:

$$(\forall u, v \in F) \varphi(uv) = \varphi(u \underline{v}).$$

Proof. Let $u, v \in F$ be such that $\text{ex}(v) = n$, i.e. $v = (\underline{v})^n$. Then:

$$\varphi(uv) = \varphi(u) \varphi(v) = \varphi(u) \varphi((\underline{v})^n) = \varphi(u) \varphi((\underline{v}))^n = \varphi(u) \varphi(\underline{v}) = \varphi(u \underline{v}). \quad \square$$

From 1.5 we obtain the following corollary:

1.6. Let $\mathbf{G} = (G, \cdot) \in \mathcal{U}_r$, $\lambda: B \rightarrow G$ and φ be the homomorphism from \mathbf{F} into \mathbf{G} which extends λ . Then the restriction ψ of φ on \mathbf{R} is a homomorphism from \mathbf{R} into \mathbf{G} , which extends λ . \square

Finally, Th. 1 is a corollary of 1.2, 1.3 and 1.6.

The following properties will be used in the next sections.

1.7. \mathbf{R} is \mathcal{U}_r -injective and (v, w) is the pair of divisors of $u \in R * R$ iff

$$|u| = |v| + |w|.$$

Proof. If $u \in R$, $k \in \mathbb{N}$, then we denote by u_*^k the k -th power of u in \mathbf{R} , i.e.

$$u_*^1 = u, \quad u_*^{k+1} = u_*^k * u. \quad (1.7)$$

By (1.6), we have: $n \geq 2 \Rightarrow |u_*^n| > |u|$, and this implies that the condition 1) from Section 0 holds. If $u \in R * R$, then $u = v * w = vw$, where $v, w \in R$ and $\underline{w} = w$. Then $u = v' * w'$ iff $v' = v$ and $\underline{w}' = \underline{w} = w$. This implies that the condition 2) of Section 0 is satisfied, as well. \square

The following two properties are also clear.

1.8. If the operation \bullet is defined in \mathbb{N} as follows:

$$(\forall m, n \in \mathbb{N}) \quad m \bullet n = m + 1, \quad (1.8)$$

then (\mathbb{N}, \bullet) is a \mathcal{U}_r -free groupoid with the basis $\{1\}$. The family of subgroupoids of (\mathbb{N}, \bullet) is infinite, and each of them is isomorphic to (\mathbb{N}, \bullet) . \square

1.9. If $\mathbf{G} = (G, \cdot) \in \mathcal{U}_r$, and $a \in G$, then the subgroupoid $\mathbf{Q} = \langle a \rangle$ of \mathbf{G} generated by a is determined as follows:

$$Q = \{a^n \mid n \in \mathbb{N}\}, \quad a^m a^n = a^{m+n}. \quad (1.9)$$

And, Q is \mathcal{U}_r -free with basis $\{a\}$ iff:

$$(\forall m, n \in \mathbb{N}) \quad (a^m = a^n \Rightarrow m = n). \quad \square \quad (1.10)$$

(As usual we say that $\langle a \rangle$ is the *cyclic subgroupoid* of \mathbf{G} , generated by a .)

2. \mathcal{U}_r -injective groupoids

Below we assume that $\mathbf{H} = (H, \cdot)$ is a \mathcal{U}_r -injective groupoid, and: $a, b, c, d \in H$, $m, n, k \in \mathbb{N}$.

Using the implication: $xy = x'y' \Rightarrow x = x'$, and the definition of the class of \mathcal{U}_r -injective groupoids, the statements that follow can be easily shown.

$$a^n = b^n \Rightarrow a = b. \quad (2.1)$$

$$a^{m+1} = b^{m+n} \Rightarrow a = b^n. \quad (2.2)$$

$$a^m = a^n \Rightarrow m = n. \quad (2.3)$$

As a corollary of **1.9** and (2.3), we obtain:

2.1. The subgroupoid $\langle a \rangle$ of \mathbf{H} , generated by $a \in \mathbf{H}$ is \mathcal{U}_r -free with the basis $\{a\}$. \square

2.2. For every $a \in H$ there is a unique pair (b, n) , such that

$$a = b^n \quad \text{and} \quad (b = c^m \Rightarrow m = 1).$$

(As in the groupoid \mathbf{F} we say that b is the *base* and n the *exponent* of a , and use the following notations: $b = \underline{a}$, $n = \text{ex}(a)$.)

Proof. Assume that there exists a pair (b, n) , such that $a = b^n$ and $n \geq 2$. Then, the right divisor c of a is the base of a . From $b^{n-1}b = b^{n-1}c$ it follows that there exists $m \in \mathbb{N}$ such that $b = c^m$, and therefore $a = (c^m)^n = c^{m+n-1}$, which implies that $\text{ex}(a) = m + n - 1$. \square

The following two statements are clear.

$$(\underline{a}^n) = \underline{a}, \quad \text{ex}(a^m) = m - 1 + \text{ex}(a). \quad (2.4)$$

$$a^m = b^n \Rightarrow \underline{a} = \underline{b}. \quad (2.5)$$

As a corollary from **2.1** and (2.5) we obtain: •

2.3. A cyclic subgroupoid $\langle a \rangle$ of H is maximal iff $\underline{a} = a$; and, any two distinct maximal cyclic subgroupoids of H are disjoint. \square

2.4. If $\underline{a} = a$, $\underline{b} = b$, $a \neq b$, $n \in \mathbb{N}$ and $c = a^n b$, then $\underline{c} = c$.

Proof. Assume that $\underline{c} = d \neq c$. Then $c = d^{m+1}$, where $m + 1 = \text{ex}(c) \geq 2$, and therefore $b = d$, $a^n = d^m$; by (2.5), $a^n = d^m$ implies $a = \underline{a} = \underline{d} = d = b$, a contradiction. \square

2.5. If the subset $A \subseteq H$ is defined by

$$A = \{\underline{a} \mid a \in H\}, \quad (2.6)$$

then A is either singleton or infinite.

Proof. If A contains at least two distinct elements, then by **2.4**, A is infinite. \square

2.6. Let ψ be the mapping from $(H \times \mathbb{N}) \times H$ into H defined by

$$\psi((a, n), b) = a^n b, \quad (2.7)$$

and

$$D = (A \times \mathbb{N}) \times A \setminus \{((a, n), a) \mid a \in A, n \in \mathbb{N}\}, \quad (2.8)$$

where A is defined in (2.6). Then the restriction φ of ψ on D is injective and $\text{im}\varphi \subseteq A$.

Proof. The inclusion $\text{im}\varphi \subseteq A$ follows from **2.4**. If $a, b, c, d \in A$, $m, n \in \mathbb{N}$ are such that $a \neq b$, $c \neq d$, $a^n b = c^m d$, then $b = d$, and $a^n = c^m$, and therefore: $a = c$, $m = n$. (Note that if A is a singleton set, then $D = \emptyset$.) \square

The last result suggests the following *construction*.

Let A be a singleton or an infinite set, and let $M = A \times \mathbb{N}$, where the equality $(a, 1) = a$, for each $a \in A$ is assumed. Let $\varphi: ((a, n), b) \mapsto \varphi((a, n), b)$ be an injection from the set (2.8) into A . Define an operation \bullet on M as follows:

$$(a, m) \bullet (a, n) = (a, m + 1), \quad (2.9)$$

$$a \neq b \Rightarrow (a, m) \bullet (b, n) = \varphi((a, m), b). \quad (2.10)$$

Denote by (A, φ) the groupoid $M = (M, \bullet)$.

The following *characterization of \mathcal{U}_r -injective groupoids* can be easily shown.

2.7. (A, φ) is a \mathcal{U}_r -injective groupoid, such that

$$A = \{(\underline{a}, n) \mid a \in A, n \in \mathbb{N}\}, \quad (2.6')$$

and $A \setminus \text{im}\varphi$ is the set of primes in (A, φ) .

Conversely, let \mathbf{H} be a \mathcal{U}_r -injective groupoid and A be defined by (2.6). Then \mathbf{H} is isomorphic to (A, φ) , where φ is the restriction on D of the mapping ψ , defined by (2.7). \square

2.8. The class of \mathcal{U}_r -free groupoids is a proper subclass of the class of \mathcal{U}_r -injective groupoids.

Proof. By 1.7, the class of \mathcal{U}_r -free groupoids is a subclass of the class of \mathcal{U}_r -injective groupoids. Let A be an infinite set. Then there exist groupoids (A, φ) such that $\text{im}\varphi = A$, and thus the set of primes in (A, φ) is empty. Therefore (A, φ) is not \mathcal{U}_r -free. \square

2.9. A groupoid (A, φ) is \mathcal{U}_r -free iff the set of primes generates (A, φ) .

Proof. If (A, φ) is \mathcal{U}_r -free, then the set of primes generates (A, φ) by Th. 1. Assume that $B = A \setminus \text{im}\varphi$ (the set of primes in (A, φ)) generates (A, φ) . By 2.1, if $B = \{b\}$ is a singleton set, (A, φ) is \mathcal{U}_r -free. It remains the case when B contains at least two distinct elements. Then, A is infinite.

Define a sequence of sets $\{B_k \mid k \geq 1\}$ as follows: $B = B_1$,

$$c \in B_{k+1} \Leftrightarrow c = \varphi((a, n), b), \quad (2.11)$$

where:

$$a \neq b, \quad n \in \mathbb{N}, \quad a \in B_i, \quad b \in B_j, \quad i, j \leq k, \quad k \in \{i, j\}. \quad (2.12)$$

The relations $B \cap \text{im}\varphi = \emptyset$, $\text{im}\varphi \subseteq A$, (2.11), (2.12), and the fact that φ is injective, imply:

$$B_{k+1} \cap (\cup\{B_i \mid 1 \leq i \leq k\}) = \emptyset, \quad (2.13)$$

and $\cup\{B_k \mid k \geq 1\} = A$, where the union is disjoint.

Let $\mathbf{G} \in \mathcal{U}_r$, and $\lambda: B \rightarrow G$. Define a set of mappings $\{\alpha_k: B_k \rightarrow G \mid k \geq 1\}$ as follows:

$$\alpha_1 = \lambda, \quad \alpha_{k+1}(d) = \alpha_i(a)^n \alpha_j(b), \quad (2.14)$$

where $d = \varphi((a, n), b) \in D_{k+1}$, $a \in B_i$, $b \in B_j$, $n \in \mathbb{N}$.

There is a unique mapping $\alpha: A \rightarrow G$ such that, for each $k \in \mathbb{N}$, α_k is the restriction of α on B_k . Finally, the mapping $\lambda: (A, \varphi) \rightarrow G$ defined by:

$$\bar{\lambda}((a, n)) = \alpha(a)^n$$

is a homomorphism which extends λ . \square

Now we can complete the proof of Th. 2.

Assume that \mathbf{H} is a \mathcal{U}_r -injective groupoid.

By 1.7, (i) \Rightarrow (ii). (Namely, if \mathbf{H} is \mathcal{U}_r -free, then it is isomorphic to the corresponding canonical \mathcal{U}_r -groupoid.) Clearly, (ii) \Rightarrow (iii).

Assume that \mathbf{H} satisfies (iii), i.e. every divisor chain in \mathbf{H} is finite. From the \mathcal{U}_r -injectivity of \mathbf{H} , it follows that any element of \mathbf{H} has at most two distinct divisors, and this, by an application of König Lemma (for example [6; 381] or [7; 4]) implies that the set of divisor chains in \mathbf{H} with the same first member a is finite. Then the last members of such maximal divisor chains are primes in \mathbf{H} and a belongs to the subgroupoid generated by them. Therefore, the set B of primes in \mathbf{H} generates \mathbf{H} . Thus (iii) \Rightarrow (iv). From 2.8 we also obtain that (iv) \Rightarrow (i). This completes the proof of Th. 2. \square

3. Subgroupoids of \mathcal{U}_r -free groupoids

The following statement is "a half" of the first part of Th. 3.

3.1. *The class of \mathcal{U}_r -injective groupoids is hereditary.*

Proof. Let \mathbf{H} be a \mathcal{U}_r -injective groupoid and \mathbf{Q} a subgroupoid of \mathbf{H} . We will show that \mathbf{Q} is \mathcal{U}_r -injective. Clearly, the condition 1), in the definition of the class of \mathcal{U}_r -injective groupoids, is hereditary, and thus it remains to show that \mathbf{Q} satisfies the condition 2).

Let $a \in \mathbf{Q}\mathbf{Q}$. Then there exist $b', c' \in \mathbf{Q}$ such that $a = b'c'$. If (b, c) is the pair of divisors of a in \mathbf{H} , then $b = b'$, and $c' = c^n$, for a (unique) $n \in \mathbb{N}$. Let k be the least positive integer such that $d = c^k \in \mathbf{Q}$. Then $k \leq n$ and $c' = d^{n-k+1}$. This implies that \mathbf{Q} satisfies the condition 2) as well. Namely, if $a \in \mathbf{Q}\mathbf{Q}$, and (b, c) is the pair of divisors of a in \mathbf{H} , then (b, c^k) is the pair of divisors of a in \mathbf{Q} . \square

In 3.2 and 3.3 we assume that \mathbf{H} is a \mathcal{U}_r -injective groupoid, and \mathbf{Q} a subgroupoid of \mathbf{H} .

3.2. *If $a \in \mathbf{Q}$, and $\underline{a}_{\mathbf{Q}}$ is the base of a in \mathbf{Q} , then there is a (unique) $k \in \mathbb{N}$ such that $\underline{a}_{\mathbf{Q}} = (\underline{a})^k$, where \underline{a} is the base of a in \mathbf{H} . \square*

3.3. *If $a \in \mathbf{Q}$ is such that $\underline{a}_{\mathbf{Q}} = a = (\underline{a})^k$, where $k \geq 2$, then a is prime in \mathbf{Q} .*

Proof. Namely, the assumption that $a = (\underline{a})^k$ is not a prime in \mathbf{Q} would imply that $(\underline{a})^{k-1} \in \mathbf{Q}$. \square

3.4. If Q is a subgroupoid of a \mathcal{U}_r -free groupoid H , then the set P of primes in Q generates Q .

Proof. By 1.7 H is \mathcal{U}_r -injective and there exists a mapping $a \mapsto |a|$ from H into \mathbb{N} such that: if $a \in HH$ and (b, c) is the pair of divisors of a in H , then $|a| = |b| + |c|$. By 3.1, Q is \mathcal{U}_r -injective, and if $a \in QQ$ and (b, c) is the pair of divisors of a in H , then there exists a (unique) $k \in \mathbb{N}$ such that (b, c^k) is the pair of divisors of a in Q .

Let m be the least positive integer such that $Q \cap \{a \mid a \in H, |a| = m\} = S$ is non-empty. Then $S \subseteq P$ and thus $P \neq \emptyset$. Denote by T the subgroupoid of Q generated by P , and assume that

$$a \in Q \ \& \ |a| \leq n \Rightarrow a \in T.$$

Let $a \in Q$ and $|a| = n + 1$. We will show that $a \in T$. Clearly, $a \in P \Rightarrow a \in T$, and thus we can assume that $a \in QQ$. Let (b, c) be the pair of divisors of a in H . By 1.7, we have $|a| = |b| + |c|$ and thus $|b|, |c| \leq n$. By the proof of 3.1, there is a (unique) $k \in \mathbb{N}$ such that (b, c^k) is the pair of divisors of a in Q , and thus $b, c^k \in Q$, and $|b| \leq n$. If $k = 1$, then $c \in Q$ as well, and thus $a \in T$. Finally, if $k \geq 2$, then by 3.3, $c^k \in P$, and therefore $a = bc^k \in T$. \square

Now we can complete the proof of the first part of Th.3.

3.5. The class of \mathcal{U}_r -free groupoids is hereditary.

Proof. Let Q be a subgroupoid of a \mathcal{U}_r -free groupoid H . By 1.7, 3.1, 3.4 and Th.2, Q is \mathcal{U}_r -free. \square

The second part of Th.3 follows from 1.8, and the third one is a corollary from the following proposition.

3.6. Let: $H = (H, \cdot)$ be a \mathcal{U}_r -free groupoid, $a, b \in H$ be such that $\underline{a} = a \neq b = \underline{b}$ and $C = \{C_k \mid k \geq 1\}$ be defined by:

$$c_1 = ab, \quad c_{k+1} = c_k b. \quad (3.2)$$

Then the subgroupoid Q generated by C is \mathcal{U}_r -free with infinite rank.

Proof. By 3.5, Q is \mathcal{U}_r -free. By induction on $m + n$, one can show that: $c_m = c_n \Leftrightarrow m = n$, and thus C is infinite. Clearly $a \notin Q, b \notin Q$, and this implies that C coincides with the set of primes in Q . (Namely, (a, b) is the pair of divisors of c_1 in H , and $a \notin Q, b \notin Q$; this implies that c_1 is a prime in Q ; assuming that c_k is a prime in Q , we obtain in the same way that c_{k+1} is also prime in Q . \square

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СЛОБОДНИ ГРУПОИДИ СО $xy^2 = xy$

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Резиме

Главните резултати во работава се Теоремите 1, 2 и 3. Во теоремата 1 се дава каноничен опис на слободните објекти во многуобразието \mathcal{U}_r од групоиди коишто го задоволуваат идентитетот $xy^2 = xy$. Во Теоремата 2 е окарактеризирана класата \mathcal{U}_r -слободни групоиди во рамките на класата \mathcal{U}_r -инјективни групоиди. На крајот, во Теоремата 3 е покажано дека секоја од споменатите класи е наследна и дека секој \mathcal{U}_r -слободен групоид со ранг 2 содржи подгрупоиди со бесконечен ранг.

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