According to *L.* 2.2, we have:

Proposition 3.3. If $a \in Q$, then there exists a positive integer $L_Q(a)$ such that $L_Q(a)$ is the largest length of divisor chains in $\mathbf{Q}$ with the first member a. Moreover: $L_Q(a) \leq L(a)$. □

We will show:

Proposition 3.4. If $b \in B$ and $\mathbf{Q}$ is generated by b^2, b^2b, then $\mathbf{Q}$ is not $\mathcal{V}$–free.

Proof. Clearly: b^2, $b^2b \in Q$, $(b^2)^2b^2 = (b^2b)^2 \in Q$ and $b^2 \neq b^2b$, but there is no $v \in Q$ such that $b^2 = v^2$. Thus $\mathbf{Q}$ does not satisfy (iv), i.e. $\mathbf{Q}$ is not $\mathcal{V}$–free. ($\{b^2, b^2b\}$ is the set of primes in $\mathbf{Q}$.) □

By *Th.* 2 and *Pr.* 3.1, we have:

Proposition 3.5. $\mathbf{Q}$ is $\mathcal{V}$–free if:

$$u \neq v,\ uv \in Q\ u^2, v^2 \in Q \Rightarrow u, v \in Q\,, \tag{3.1}$$

for any u, $v \in H$. □

As a concequence of *Pr.* 3.5 we have:

Propositon 3.6. Each of the following conditions is sufficient for $\mathbf{Q}$ to be $\mathcal{V}$–free:

$$x^2 \in Q \Rightarrow x \in Q\,, \tag{3.2}$$

$$u \neq v,\ uv \in Q \Rightarrow u,\ v \in Q\,.\ \square \tag{3.3}$$

The following property will help to complete the proof of *Th.* 3.

Propositon 3.7. If $[u] = 0$ for every prime in $\mathbf{Q}$, then $\mathbf{Q}$ is $\mathcal{V}$–free.

Proof. It is enough to show that $\mathbf{Q}$ satisfies the condition (3.2) and this can be shown by induction on $L_Q(x^2)$. □

Propositon 3.8. Let $b \in B$, $a_1 = b^2b$, $a_{k+1} = a_kb$, $A = \{a_k \mid k \geq 1\}$. If $\mathbf{Q}$ is generated by A, then $\mathbf{Q}$ is $\mathcal{V}$–free with an infinite rank.

Proof. All the elements of A are primes in $\mathbf{Q}$, and then apply *Pr.* 3.7. □

Propositon 3.9. Let $C = \{b^2\} \cup A$, where A is as in *Pr.* 3.8. If $\mathbf{S}$ is the groupoid generated by C, then $\mathbf{S}$ is not $\mathcal{V}$–free, and all the elements of C are primes in S. □