By (0.5) and (1.3) we obtain:

$$|u * v| = 2^m \left|u^{(-m)} v^{(-m)}\right| = 2^m \left(2^{-m}|u| + 2^{-m}|v|\right) = |u| + |v|, \tag{1.10}$$

i.e the restriction of the norm on R is a homomorphism from $\mathbf{R}$ onto the additive groupoid of positive integers. This restriction will be called the **norm on R**.

Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}$, $\lambda: B \to G$ be an arbitrary mapping, and $\varphi: \mathbf{F} \to \mathbf{G}$ be the homomorphism which is an extension of λ. Denote by ψ the restriction of φ on R. If $u, v \in R$ and $m = \min\{[u], [v]\}$, then:

$$\begin{aligned}
\psi(u * v) &= \varphi\left(\left(u^{(-m)} v^{(-m)}\right)^{(m)}\right) = \left(\varphi\left(u^{(-m)} v^{(-m)}\right)\right)^{(m)} \\
&= \left(\varphi\left(u^{(-m)}\right) \varphi\left(v^{(-m)}\right)\right)^{(m)} = \\
&= \left(\varphi\left(u^{(-m)}\right)\right)^{(m)} \left(\varphi\left(v^{(-m)}\right)\right)^{(m)} \\
&= \varphi\left(\left(u^{(-m)}\right)^{(m)}\right) \varphi\left(\left(v^{(-m)}\right)^{(m)}\right) \\
&= \varphi(u)\, \varphi(v) \\
&= \psi(u)\, \psi(v),
\end{aligned}$$

i.e. $\psi: \mathbf{R} \to \mathbf{G}$ is a homomorphism.

Thus $\mathbf{R}$ is a free groupoid in $\mathcal{V}$, with a basis B. B is the unique basis of $\mathbf{R}$, for it is a subset of any generating subset of $\mathbf{R}$. This completes the proof of *Th.* 1.

Remark. The above proof of *Th.* 1 is almost a direct consequence of the previously given definitions and results. Of course, some more general results could be used, but they would make the corresponding proof even more complicated. We will not include here discussions of that kind, and the interested reader is addressed to the corresponding books and papers (for example: [2], III.5; [4], §10; [5], 1.4; [6], 2.9).

2. An axiom system for $\mathcal{V}$-free groupoids

The main object of this section is the proof of **Th. 2.**

Proposition 2.1. Every $\mathcal{V}$-free groupoid satisfies the conditions (i)–(iv) of **Th. 2.**

Proof. By **Th. 1.**, it is enough to show that $\mathbf{R} = (R, *)$ satisfies (i)–(iv).

Below we assume that $x, y, z, u, v \in R$.