Then the set B of primes in $\mathbf{F}$ is nonempty and it is the unique basis of $\mathbf{F}$.

If $\mathbf{G} = (G, \cdot)$ is a given groupoid, then for any nonnegative integer k we define a transformation (k): $x \mapsto x^{(k)}$ of G in the following way:

$$x^{(0)} = x, \qquad x^{(k+1)} = x^{(k)}x^{(k)}. \tag{0.1}$$

From the condition a) we obtain that, in a free groupoid $\mathbf{F}$, (k) is injective, for any $k \geq 0$. Thus, for each $k \geq 0$, there exists an injective partial transformation $(-k)$: $x \mapsto x^{(-k)}$ defined in $\mathbf{F}$ as follows:

$$y^{(-k)} = x \Leftrightarrow y = x^{(k)}. \tag{0.2}$$

For any $u \in F$, there exists the largest integer $[u] = m$, such that $u^{(-m)} \in F$. (The integer $[u]$ will be called the **exponent** of u in F.)

The following subset R of F will play an important role in the paper. Namely, if B is the basis of $\mathbf{F}$, then we define R as the least subset of F such that $B \subseteq R$, and if $u = vw \in F\backslash B$, then:

$$u \in R \Leftrightarrow [v,\, w \in R \quad \text{and} \quad (v = w \text{ or } \min\{[v],\, [w]\} = 0)]. \tag{0.3}$$

Recall that we denoted by $\mathcal{V}$ the variety of groupoids which satisfy the law

$$(xy)^2 = x^2y^2. \tag{0.4}$$

If $\mathbf{G} \in \mathcal{V}$, then we call $\mathbf{G}$ a **$\mathcal{V}$-groupoid**, and if it is free in $\mathcal{V}$, we say that it is **$\mathcal{V}$-free**.

Now we are ready to state the main results.

Theorem 1. If $u,\, v \in R$, $m = \min\{[u],\, [v]\}$ and $u * v$ is defined by:

$$u * v = \left(u^{(-m)}v^{(-m)}\right)^{(m)}, \tag{0.5}$$

then $\mathbf{R} = (R, *)$ is a $\mathcal{V}$-free groupoid and the set B (i.e. the basis of F) is the unique basis of $\mathbf{R}$.

Theorem 2. A $\mathcal{V}$-groupoid $\mathbf{H} = (H, \cdot)$ is $\mathcal{V}$-free iff the following conditions hold

(i) Every divisor chain in $\mathbf{H}$ is finite.

(ii) $x^2 = y^2 \Rightarrow x = y$.

(iii) $xy = uv,\ x \neq y,\ u \neq v \Rightarrow x = u,\ y = v.$ [2)]

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[2] "$p, q, \ldots$" means "$p \& q \& \ldots$".