Define $x^{(k)}$, for $k \geq 0$, by:

$$x^{(0)} = x\,, \qquad x^{(k+1)} = \left(x^{(k)}\right)^n. \tag{5.3}$$

Note that (5.3), for $n = 2$, coincides with (0.1).

As before, $\mathbf{F} = (F, \cdot)$ denotes a free groupoid with the basis B. Since the implication

$$x^k = y^m \Rightarrow x = y\,, \quad k = m \tag{5.4}$$

is true in $\mathbf{F}$, the mapping $x \mapsto x^{(m)}$ is an injective transformation of F. Thus we can define $x^{(-k)}$ and $[x]$, as in the special case $n = 2$.

It is easy to show that (1.2)–(1.6), *L.* 1.1 and *L.* 1.2 are true for any $n \geq 2$.

Now we will define F_n as the least subset of F such that $B \subseteq F_n$ and: $vw \in F_n \Leftrightarrow$

$$\left[\left(w \in F_n,\ v = w^{n-1}\right) \text{ or } \left(v,\, w \in F_n \text{ and } \min\{[v], [w]\} = 0\right)\right]. \tag{5.5}$$

Therefore, $F_2 = R$ where R is defined by (0.3). Note that the implication $vw \in F_n \Rightarrow v,\, w \in F_n$, for $n \geq 3$, is not true. (For example, if $b \in B$ and $n = 3$, then $b^{(2)} = \left(b^3\right)^2 \cdot b^3 \in F_3$, but $\left(b^3\right)^2 \notin F_3$.)

The following statement is a generalization of *Th.* 1.

Theorem 1′. $\mathbf{F}_n = (F_n, *)$ is a $\mathcal{V}_n$–free groupoid with the unique basis B. Here:

$$u * v = \left(u^{(-m)}\, v^{(-m)}\right)^{(m)},$$

where $u,\, v \in F_n$ and $m = \min\{[u], [v]\}$. □.

This generalization is obtained by substituing R by F_n. The situation with the other theorems is similar, except with *Th.* 4. Namely, the definition of the operation (k), given by (0.6), does make sense for $n \geq 3$ also, but it is easy to show that $\mathbf{F}_n^{(k)} \notin \mathcal{V}_n$, for $n \geq 3$.

The statements (ii), (iii) and (iv) of *Th.* 2, in the formulation of *Th.* 2′ (besides the substitution of $\mathcal{V}$ by $\mathcal{V}_n$), obtain the following forms:

(ii′) $x^n = y^n \Rightarrow x = y$.

(iii′) $xy = uv,\ \ x \neq y^{n-1},\ \ u \neq v^{n-1} \Rightarrow x = u,\ \ y = v$.

(iv′) $x^n = yz,\ \ y \neq z^{n-1} \Rightarrow (\exists u,\, v) x = uv,\ \ y = u^n,\ \ z = v^n$.

According to *Th.* 3′, note that if $\mathbf{H}$ is a $\mathcal{V}_n$–free groupoid and if $\mathbf{Q}$ is the subgroupoid of $\mathbf{H}$ generated by $A = \{a_p | p \geq 1\}$, where $a_p = b^{n+p}$ (b is an element of the basis B), then A is the basis of $\mathbf{Q}$. Therefore, $\mathbf{Q}$ has an infinite rank. If $\mathbf{S}_p$