Suppose that different γ, δ with the mentioned property do not exist and put $x = \alpha_0$. Then there exists a (unique) $\alpha_1 \in Q$ such that $\alpha_0 = \alpha_1^{(k+1)}$. Let the sequence $\alpha_0, \alpha_1, \dots, \alpha_i, \dots$ be such that $\alpha_i = \alpha_{i+1}^{(k+1)}$, for every i. Since $|\alpha_i| > |\alpha_{i+1}|$, the sequence is finite. Let α_n be its last member. By the definition of the sequence we have:

$$\alpha_0 = y^{(-1)}\, z^{(-1)}\,, \qquad \alpha_1 = y^{(-k-2)}\, z^{(-k-2)}\,,$$
$$\alpha_2 = y^{(-2k-3)}\, z^{(-2k-3)}\,, \dots, \qquad \alpha_n = y^{(-nk-n-1)}\, z^{(-nk-n-1)}\,.$$

By the last equlity, there exist u, $v \in Q$ such that $\alpha_n = u(k)v = (uv)^{(k)}$. Clearly, $u \neq v$ (since α_n is the last member of the sequence). From the equality $(uv)^{(k)} = y^{(-nk-n-1)}\, z^{(-nk-n-1)}$ we have:

$$u = y^{(-nk-n-k-1)}\,, \qquad v = z^{(-nk-n-k-1)}\,, \qquad \text{i.e.}$$
$$y = u^{(nk+n+k+1)}\,, \qquad z = v^{(nk+n+k+1)}\,.$$

Therefore:
$$x = \alpha_0 = y^{(-1)}\, z^{(-1)} = u^{(nk+n+k)}\, v^{(nk+n+k)} =$$
$$= \left(u^{(nk+n)}\, v^{(nk+n)}\right)^{(k)} = (\gamma\delta)^{(k)}\,,$$

where $\gamma = u^{(nk+n)} \neq v^{(nk+n)} = \delta$. □

Propositon 4.8. **Q** is $\mathcal{V}$–free.

Proof. Let x, y, $z \in Q$ be such that $x^2 = uz$, $y \neq z$. By *L.* 4.7, there exist γ, $\delta \in Q$ such that $\gamma \neq \delta$, $x = (\gamma\delta)^{(k)}$. By $x^2 = yz$ it follows that $\gamma^{(k+1)}\,\delta^{(k+1)} = yz$, i.e $y = \gamma^{(k+1)}$, $z = \delta^{(k+1)}$. Therefore: $x = \gamma(k)\delta$, $y = \gamma(k)\gamma$, $z = \delta(k)\delta$. Thus **Q** satisfies the condition (iv), and from *Pr.* 4.6 it follows that **Q** satisfies (i), (ii), (iii) as well. □

5. *On the equation* $(xy)^n = x^n y^n$

Denote by $\mathcal{V}_n$ the variety of groupoids which satisfy the identity

$$(xy)^n = x^n y^n\,, \tag{5.1}$$

where n is a given positive integer. Here the powers are defined in the usual way, i.e. by:

$$x^1 = x\,, \qquad x^{k+1} = x^k x\,. \tag{5.2}$$

By the above considerations, $\mathcal{V}_1$ is the variety of all groupoids, and $\mathcal{V}_2$ the variety $\mathcal{V}$. Further on we assume that n is a fixed integer and $n \geq 2$.