## 4. Some properties of the functors $(k)$ in $\mathcal{V}$

The proofs of the following three statements are obvious.

**Propositon 4.1.** $\mathbf{G} \in \mathcal{V} \Rightarrow G^{(k)} \in \mathcal{V}$. □

**Propositon 4.2.** If $\mathbf{G} = (G, \cdot)$, $\mathbf{S} = (S, \cdot) \in \mathcal{V}$ and $\varphi: G \to S$ is a homomorphism from $\mathbf{G}$ into $\mathbf{S}$, then $\varphi: \mathbf{G}^{(k)} \to \mathbf{S}^{(k)}$ is a homomorphism from $\mathbf{G}^{(k)}$ into $\mathbf{S}^{(k)}$ as well. □

Thus for every $k \geq 0$, $(k)$ is a functor in $\mathcal{V}$.

**Propositon 4.3.** If $k$, $n \geq 0$ and $\mathbf{G} \in \mathcal{V}$, then $\left(\mathbf{G}^{(k)}\right)^{(n)} = \mathbf{G}^{(kn+n)}$. □

Below we assume that $\mathbf{H}$ is a $\mathcal{V}$–free groupoid, with the basis $B$, and that $k$ is a positive integer. The subgroupoid of $\mathbf{H}^{(k)}$ generated by $B$ will be denoted by $\mathbf{Q}$. Also (i), (ii), (iii) and (iv) are the conditions stated in *Th.* 2.

The following statements 4.4–4.6 are obvious or they can be easily shown.

**Propositon 4.4.** If $x$, $y$, $u$, $v \in H$, then:

$$x(k)y = u(k)v \Leftrightarrow xy = uv\,. \quad \Box$$

**Propositon 4.5.** If $k \geq 1$, then $B$ is a proper subset of the set $P$ of primes in $\mathbf{H}^{(k)}$.

(Each element $b \in B$ is prime in $\mathbf{H}^{(k)}$, and for every $u \in H$, $b \in B$, we have $ub \in P$, $ub \notin B$.) □

**Propositon 4.6.** $\mathbf{H}^{(k)}$ satisfies (i), (ii) and (iii) of *Th.* 2, but for $k \geq 1$, $\mathbf{H}^{(k)}$ is not $\mathcal{V}$–free.

(Namely, if $b \in B$, then: $b^2b\,(k)\,b^2b = (b^2)^2\,(k)\,b^2$, $(b^2)^2 \neq b^2$, but $b^2$ is a prime in $\mathbf{H}^{(k)}$, for $k \geq 1$. Thus $\mathbf{H}^{(k)}$ does not satisfy (iv).) □

In order to complete the proof of *Th.* 4, first we will show the following

**Lemma 4.7.** If $x$, $y$, $z \in Q$, $y \neq z$ and $x^2 = yz$, then there exist $\gamma$, $\delta \in Q$ such that $\gamma \neq \delta$ and $x = (\gamma\delta)^{(k)}$.

**Proof.** The equality $x^2 = yz$ implies that $[yz] \geq 1$, and (by (2.5)) we have $[y]$, $[z] \geq 1$ and $x = (yz)^{(-1)} = y^{(-1)}\,z^{(-1)}$. Thus: $x \in Q \backslash B$, and so there exist $\gamma$, $\delta \in Q$ such that $x = \gamma(k)\delta = (\gamma\delta)^{(k)}$. It remains to show that there exist different $\gamma$, $\delta$ with the above property.