

FREE OBJECTS IN SOME VARIETIES OF GROUPOIDS

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Abstract

We give a canonical description of free objects in the variety $\mathcal{V}_{m,n}$ of groupoids which satisfy the law $x^m y^n = z_1 z_2 \dots z_m$, where $z_i = x$ if i is odd, $z_j = y$ if j is even, and m, n are integers such that $m > n \geq 2$. We also consider a derived quasivariety $\mathcal{V}_{m,n}^\square$ of groupoids in which only trivial identities hold.

0. Introduction

A *groupoid* is an algebra $\mathbf{G} = (G, \cdot)$ with one binary operation $(x, y) \mapsto x \cdot y$. As usual, the symbol of the operation and some brackets will be omitted. Namely, if $a, a_1, a_2, \dots, a_k, a_{k+1} \in G$, then:

$$a^1 = a, \quad a^{k+1} = a^k a, \quad (0.1)$$

$$a_1 a_2 \dots a_k a_{k+1} = (a_1 a_2 \dots a_k) a_{k+1}. \quad (0.2)$$

If k is a positive integer and $a, b, c \in G$, $x_i = a$, $x_j = b$ for $1 \leq i, j \leq k$, where i is odd and j is even, then:

$$\begin{aligned} \underline{abk} &= x_1 x_2 \dots x_k, & \underline{ab1} &= a, \\ c \underline{abk} &= c x_1 \dots x_k, & c \underline{ab1} &= ca, & c \underline{ab0} &= c. \end{aligned} \quad (0.3)$$

Note that if $a, b, a_\nu, b_\nu \in G$, $k > 0$, $k_1, k_2, \dots \geq 0$, then

$$\underline{abk} \underline{a_1 b_1 k_1} \dots \underline{a_j b_j k_j} \quad (0.4)$$

is an element of G which is defined by:

$$\begin{aligned} \underline{abk} \underline{a_1 b_1 0} &= \underline{abk}, & \underline{abk} \underline{a_1 b_1 1} &= (\underline{abk}) \cdot a_1, \\ \underline{abk} \underline{a_1 b_1 2} &= ((\underline{abk})a_1)b_1, \dots \end{aligned}$$

It should be pointed out that, in (0.4), \underline{abk} is an element of G , and each of the triples $\underline{a_i b_i k_i}$ takes part as a sequence of elements where the multiplication is "from left to the right" according to the definition (0.2). For example: $\underline{ab3}, \underline{cd2} \in G$, but $\underline{ab3} \underline{cd2} \neq (\underline{ab3}) \cdot (\underline{cd2})$. Namely,

$$\begin{aligned} \underline{ab3} \underline{cd2} &= (((ab)a)c)d = abacd, \\ (\underline{ab3}) \cdot (\underline{cd2}) &= ((ab)a)(cd). \end{aligned}$$

Recall that $\mathcal{V}_{m,n}$ is the variety of groupoids which satisfy the identity

$$x^m y^n = \underline{xym}, \quad (0.5)$$

where m, n are positive integers. Further on we will assume that $m > n \geq 2$, if it is not stated otherwise.

For every $p \geq 0$, we define transformations $x \mapsto x^{<p>}$ and $x \mapsto x^{(p)}$ of G in the following way:

$$x^{<0>} = x^{(0)} = x, \quad x^{<p+1>} = (x^{<p>})^m, \quad x^{(p+1)} = (x^{(p)})^n. \quad (0.6)$$

Clearly:

$$(x^{<p>})^{<q>} = x^{<p+q>}, \quad (x^{(p)})^{(q)} = x^{(p+q)}, \quad (0.7)$$

for all $p, q \geq 0$.

Let $\mathbf{Q} = (Q, \circ)$ and $\mathbf{G} = (G, \cdot)$ be groupoids such that $Q \subseteq G$. Q is said to be an (m, n) -subgroupoid of \mathbf{G} iff $a \circ b = a^m b^n$ for all $a, b \in Q$. The class of groupoids which are (m, n) -subgroupoids of groupoids in $\mathcal{V}_{m,n}$ will be denoted by $\mathcal{V}_{m,n}^{\square}$. So $\mathcal{V}_{m,n}^{\square}$ is derived from $\mathcal{V}_{m,n}$. ([3], III.7.)

A free groupoid (in the variety of all groupoids) with a given basis B will be denoted by $\mathbf{F} = (F, \cdot)$.

We denote by $R_{m,n}$ the least subset of F such that $B \subset F$ and

$$xy \in R_{m,n} \Leftrightarrow [x, y \in R_{m,n} \text{ and } (\forall \alpha, \beta \in F)(x \neq \alpha^m \text{ or } y \neq \beta^n)]. \quad (0.8)$$

(Further on we will write R instead of $R_{m,n}$.)

Below we define a mapping $*$: $R^2 \rightarrow F$.

Let $x, y \in R$ be such that $xy \in R$ and $[y]_n = r^{(1)}$, $y = z^{(r)}$. Then, we define $x * y$, $x^{<1>} * y^{(1)}$ and $x^{<p+1>} * y^{(p+1)}$, where $p \geq 1$, as follows.

$$x * y = xy. \quad (0.9)$$

$$x^{<1>} * y^{(1)} = \begin{cases} y^3, & \text{if } x = y^2, n = 2, m = 3 \\ z^{n+2} \frac{z^{(1)} z_{n-3} \dots z^{(r+1)} z^{(r)}_{n-3},}{\text{if } x = y^n, n \geq 3, m = n + 1} \\ \underline{xy^m}, & \text{if } x \neq y^n \text{ or } m > n + 1. \end{cases} \quad (0.10)$$

$$x^{<p+1>} * y^{(p+1)} = \left(x^{<1>} * y^{(1)} \right) \underline{x^{<1>} y^{(1)}_{m-2} \dots x^{<p>} y^{(p)}_{m-2}} \quad (0.11)$$

The following theorems are the main results in the paper.

THEOREM 1. $u * v \in R$, for all $u, v \in R$ and the groupoid $\mathbf{R} = (R, *)$ is free in $\mathcal{V}_{m,n}$ with the (unique) basis B .

THEOREM 2. $\mathcal{V}_{m,n}^{\square}$ is a proper quasi-variety of groupoids, and only trivial identities hold in $\mathcal{V}_{m,n}^{\square}$.

REMARK. (m, n) -subgroupoids are special kinds of t -subgroupoids, where $t = t(x, y)$ is a groupoid term in which two variables x, y appear. ($\mathbf{Q} = (Q, \circ)$ is a t -subgroupoid of a groupoid $\mathbf{G} = (G, \cdot)$ iff $Q \subseteq G$ and

$$a \circ b = t_{\mathbf{G}}(a, b), \quad (0.12)$$

for all $a, b \in Q$; the right-hand side of (0.12) is the value of the term $t(x, y)$ in \mathbf{G} for $x = a, y = b$.) If \mathcal{V} is a variety of groupoids, then the class of t -subgroupoids in \mathcal{V} will be denoted by \mathcal{V}^t .

The paper [7] consider a question which can be "translated" in the language of groupoids in the following way: "Is the condition «Only trivial identities hold in \mathcal{V}^t » sufficient for the class \mathcal{V}^t to coincide with the variety of all groupoids?" The answer (which follows by *Th. 2*) is negative. The question: whether the same is true for generalized subalgebras of algebras of any type Ω , remains open.

Th. 1, Th. 2 are proved in §1, §2 respectively. The obtained canonical description of free groupoids in \mathcal{V} (in *Th. 1*) is due to the fact that the rewriting system on F induced by elementary transformations

) $[y]_n$ is the largest non-negative integer r such that $y = z^{(r)}$ for some $z \in F_n$. (See also below, after (1.4.2).)

$u^m v^n \rightarrow uv^m$ is a terminating Church–Rose system. This conclusion does not hold in the case $2 \leq m \leq n$ or $m > n = 1$, which is shown in §3.

1. A canonical description of free groupoids in $\mathcal{V}_{m,n}$

In this section we will prove *Th. 1* in the case $m > n \geq 2$; first we state some properties of \mathbf{F} .

The following two properties are characteristic for a free groupoid \mathbf{F} with the basis B ([1], 1.1).

a) $ab = cd \Rightarrow a = c, b = d$.

(Any groupoid with this property is said to be *injective*.)

b) B is the set of primes in \mathbf{F} and it generates \mathbf{F} .

(An element $c \in G$ is *prime* in a groupoid $\mathbf{G} = (G, \cdot)$ iff $c \neq xy$, for all $x, y \in G$.)

The *norm* in \mathbf{F} is the homomorphism $x \mapsto |x|$ from \mathbf{F} into the additive groupoid of positive integers which is an extension of the mapping $B \rightarrow \{1\}$. Thus:

$$|b| = 1, \quad |uv| = |u| + |v|, \quad (1.1)$$

for $b \in B, u, v \in F$.

The statements below are direct consequences of (1.1) and the injectivity of \mathbf{F} . Here: $x_\nu; y_\nu, x, y, \alpha \in F, i, j, k \geq 1, p, q, r \geq 0$.

$$|x^i| = i|x|, \quad |x^{<p>}| = m^p|x|, \quad |x^{(p)}| = n^p|x|. \quad (1.2)$$

$$x^i = y^j \Rightarrow x = y, \quad i = j; \quad (1.3.1)$$

$$\begin{aligned} x_1 x_2 \dots x_i y_1 \dots y_j = z z_1 \dots z_j &\Rightarrow \\ \Rightarrow z = x_1 \dots x_i, \quad z_1 = y_1, \dots, z_j = y_j; & \end{aligned} \quad (1.3.2)$$

$$x^{<p>} = y^{<p+q>} \Rightarrow x = y^{<q>}, \quad x^{(p)} = y^{(p+q)} \Rightarrow x = y^{(q)}. \quad (1.3.3)$$

$$2 \leq k \leq m, \quad x_1 \neq x_i \text{ for some } i > 1 \Rightarrow x_1 x_2 \dots x_k \neq \alpha^m; \quad (1.4.1)$$

$$2 \leq k < m, \quad y_i \neq y_j \text{ for some } i \neq j \Rightarrow x_1 \dots x_p y_1 \dots y_k \neq \alpha^m; \quad (1.4.2)$$

According to (1.2), for any $u \in F$, there exists the largest non-negative integer k such that $u = x^{<k>}$, for some $x \in F$. This integer k will be denoted by $[u]_m$. One defines $[u]_n$ in the same way. Next, $[u, v]$ is defined by:

$$[u, v] = \min \{[u]_m, [u]_n\}. \quad (1.5)$$

By (1.5), it follows:

$$[u^{<p>}, v^{(p)}] = p + [u, v]. \quad (1.6)$$

The definition (0.8) can obtain now the following form:

$$uv \in R_{m,n} \Leftrightarrow (u, v \in R_{m,n}, [u, v] = 0). \quad (1.7)$$

(As above, we will write R instead of $R_{m,n}$.)

The following properties are also consequences of (1.7) and (1.2)–(1.6).

$$1 \leq k \leq m, x \in R \Rightarrow x^k \in R, x^{<p>}, x^{(p)} \in R. \quad (1.8)$$

$$x^{<p>} \in R \text{ or } x^{(p)} \in R \Rightarrow x \in R. \quad (1.9)$$

$$x \in R \Rightarrow (x^{m+1} \in R \Leftrightarrow [x]_n = 0). \quad (1.10)$$

$$p \geq 1, x, y \in R \Rightarrow xy^{<p>} \in R. \quad (1.11)$$

$$p \geq 1, x, y \in R \Rightarrow (xy^{(p)} \in R \Leftrightarrow [x]_m = 0). \quad (1.12)$$

$$xy \in R \Rightarrow (xyx \in R \Leftrightarrow (x \neq y^n \text{ or } m > n + 1)). \quad (1.13)$$

Assume now that $u * v \in F$ is defined by (0.9), (0.10) and (0.11), where $u, v \in R$ are such that $[u, v] = p$, $u = x^{<p>}$, $y = y^{(p)}$, $[y]_n = r$, $y = z^{(r)}$. We have to show that $u * v \in R$.

If $p = 0$, then $u * v = uv \in R$, and thus we can assume that $p \geq 1$. Consider first the case $x \neq y^n$ or $m > n + 1$. Then $x^{<1>} * y^{(1)} = xym$. By (1.13) we have $xyx = xy3 \in R$, and thus we can assume that $m \geq 4$. Then from (1.4.1) it follows: $xyxy \in R$. In the same way one can obtain: $x^{<1>} * y^{(1)} = xym \in R$. Assume now that $p \geq 1$. Then:

$$x^{<p+1>} * y^{(p+1)} = \underline{xym} \underline{x^{<1>}y^{(1)}m-2} \dots \underline{x^{<p>}y^{(p)}m-2}. \quad (1.14)$$

We will consider only the case $m = 4$ (and $n = 2$ or $n = 3$). Then:

$$x^{<p+1>} * y^{(p+1)} = xyxy x^{<1>}y^{(1)} \dots x^{<p>}y^{(p)}. \quad (1.14')$$

From (1.11) we obtain $xyxyx^{<1>} \in R$, and then (1.4.2) implies:

$xyxyx^{<1>}y^{(1)} \in R$. Continuing in this way we would get

$x^{<p+1>} * y^{(p+1)} \in R$.

It remains the case $x = y^n$, $m = n + 1$.

If $n = 2$, then $x^{<1>} * y^{(1)} = y^3 \in R$, and therefore $x^{<p+1>} * y^{(p+1)} = y^3 x^{<1>} \dots x^{<p>} \in R$, by (1.11). Thus we can assume that $n \geq 3$. In the case $n = 3$, we have $x^{<1>} * y^{(1)} = z^5 \in R$, by (1.10), and then (in the same way as in the case $m = 4$, $x \neq y^2$) one can show that $x^{<p+1>} * y^{(p+1)} \in R$, in the case $m = n + 1 = 4$, $x = y^3$, as well. It remains the case $n + 1 = m \geq 5$, $y = x^n$. Then, we obtain $x^{<1>} * y^{(1)} \in R$, by applications of (0.10), (1.10)

and (1.4.2). Finally, in the same way as in the first considered case ($x \neq y^n$ or $m > n + 1$) one can obtain that $x^{<p+1>} * y^{(p+1)} \in R$.

Thus we have the following:

PROPOSITION 1.1. $\mathbf{R} = (R, *)$ is a groupoid. \diamond

Below we will show that $(R, *) \in \mathcal{V}_{m,n}$. First, denote by u_*^k ($u \in R, k \geq 1$) the corresponding k -th power of u in \mathbf{R} , i.e.

$$u_*^1 = u, \quad u_*^{k+1} = u_*^k * u.$$

By (0.9) and (1.8): $k \leq m \Rightarrow u_*^k = u^k$, and thus:

$$u_*^m = u^m, \quad u_*^n = u^n, \quad u_*^{<p>} = u^{<p>}, \quad u_*^{(p)} = u^{(p)}, \quad (1.15)$$

for all $u \in R, p \geq 0$. This implies:

$$u_*^m * v_*^n = u^m * v^n = x^{<p+1>} * y^{(p+1)}, \quad (1.16)$$

where $u, v \in R, [u, v] = p, u = x^{<p>}, v = y^{(p)}$.

If $u, v \in R$, then $*uvm$ will be an abbreviation for the product $z_1 * z_2 * \dots * z_m$, where $z_i = u$ when i is odd, and $z_j = v$ when j is even. (Note that $uvm \in F, *uvm \in R$ and it is possible $*uvm \neq uvm$.)

From (0.9), (0.10) and (0.11) we obtain

$$\begin{aligned} *xym &= x^{<1>} * y^{(1)}, \\ *x^{<p>}y^{(p)}m &= \left(x^{<1>} * y^{(1)}\right) \underline{x^{<1>}y^{(1)}m-2} \dots \underline{x^{<p>}y^{(p)}m-2}. \end{aligned} \quad (1.17)$$

(For example, if $(m = 4, n = 2)$ or $(m = 4, n = 3, x \neq y^3)$, then:

$$*xy4 = ((x * y) * x) * y = xyxy = x^{<1>} * y^{(1)}.$$

If $p \geq 1$, then

$$\begin{aligned} *x^{<p>}y^{(p)}4 &= ((x^{<p>} * y^{(p)}) * x^{<p>}) * y^{(p)} \\ &= xyxyx^{<1>}y^{<1>} \dots x^{<p>}y^{(p)} \\ &= (x^{<1>} * y^{(1)}) x^{<1>}y^{(1)} \dots x^{<p>}y^{(p)}. \end{aligned}$$

In the case $n + 1 = m = 4, x = y^3, [y]_n = r, y = z^{(r)}$, we have:

$$*xy4 = ((x * y) * x) * y = z^5 = x^{<1>} * y^{(1)},$$

$$\begin{aligned} \underline{*x^{<p>}y^{(p)}} &= ((x^{<p>} * y^{(p)}) * x^{<p>}) * y^{(p)} \\ &= z^5 x^{<1>} y^{(1)} \dots x^{<p>} y^{(p)} \\ &= (x^{<1>} * y^{(1)}) x^{<1>} y^{(1)} \dots x^{<p>} y^{(p)}. \end{aligned}$$

So, the following equation holds:

$$u^m * v^n = \underline{*uvm}, \tag{1.18}$$

and therefore we obtain:

PROPOSITION 1.2. $R \in \mathcal{V}_{m,n}$. \diamond

The following statement "inspired" the definition of R and $*$, and it will be used in the proof of *Pr.* 1.4, as well.

PROPOSITION 1.3. If $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$, then the following implications hold:

- a) $p \geq 1 \Rightarrow x^{<p+1>} y^{(p+1)} = \underline{xym x^{<1>} y^{(1)} m - 2 \dots x^{<p>} y^{(p)} m - 2}$.
- b) $m = 3 = n + 1 \Rightarrow y^3 y^2 = y^3$.
- c) $m = 4 = n + 1, r \geq 0 \Rightarrow (z^{(r+1)})^4 z^{(r+1)} = z^5$.
- d) $m = n + 1 \geq 5, r \geq 0 \Rightarrow (z^{(r+1)})^m z^{(r+1)} = \underline{z^{m+1} z^{(1)} z m - 2 \dots z^{(r+1)} z^{(r)} m - 2}$. \diamond

PROPOSITION 1.4. If $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$ and $\varphi: F \rightarrow G$ is a homomorphism from \mathbf{F} into \mathbf{G} then the restriction ψ of φ on R is a homomorphism from \mathbf{R} into \mathbf{G} .

Proof. By using *Pr.* 1.3 and the definition of $*$. \diamond

As a consequence we obtain *Th.* 1, i.e. the following

PROPOSITION 1.5. \mathbf{R} is free in $\mathcal{V}_{m,n}$ with the (unique) basis B .

Proof. First, by the definition of R and $*$, B is the set of primes in \mathbf{R} and B generates \mathbf{R} . Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$ and $\lambda: B \rightarrow G$ be a mapping. If φ is the homomorphism from \mathbf{F} into \mathbf{G} which extends λ , then by *Pr.* 1.4, the restriction ψ of φ on R is a homomorphism from \mathbf{R} into \mathbf{G} . \diamond

Below we show a variant of *Th.* 1 concerning the variety $\mathcal{V}_{1,n}$.

PROPOSITION 1.6. $\mathcal{V}_{1,1}$ is the variety of left-zero groupoids. \diamond

PROPOSITION 1.7. If $m = 1, n \geq 2$, then $[u]_n \leq 1$ for every $u \in R (= R_{1,n})$. If an operation $*$ is defined by:

$$u * v = \begin{cases} uv, & \text{if } [v]_n = 0, \\ u, & \text{if } [v]_n = 1, \end{cases} \tag{1.19}$$

then $\mathbf{R} = (R, *)$ is a groupoid which is free in $\mathcal{V}_{1,n}$ with the basis B .

Proof. It is clear by (0.8) that $[v]_n \geq 2$ implies $v \notin R$. Therefore, by (1.19), we obtain:

$$v_*^n = \begin{cases} v^n, & \text{if } [v]_n = 0, \\ v, & \text{if } [v]_n = 1, \end{cases}$$

and so:

$$u * v_*^n = \begin{cases} u * v^n = u, & \text{if } [v]_n = 0, \\ u * v = u, & \text{if } [v]_n = 1. \end{cases}$$

Thus, $\mathbf{R} \in \mathcal{V}_{1,n}$. \diamond

2. Some properties of the class $\mathcal{V}_{m,n}^\square$

Given a groupoid $\mathbf{H} = (H, \cdot)$, by \mathbf{H}^\square will be denoted the groupoid (H, \square) defined by

$$a \square b = a^m b^n \quad (2.1)$$

(The right-hand side of (2.1) has the usual meaning in \mathbf{H} .)

In *Pr.* 2.1–*Pr.* 2.5, m, n are (arbitrary) positive integers.

PROPOSITION 2.1. $\mathbf{G} \in \mathcal{V}_{m,n}^\square$ iff there exists a groupoid $\mathbf{H} \in \mathcal{V}_{m,n}$ such that \mathbf{G} is a subgroupoid of \mathbf{H}^\square . \diamond

Propositions 2.2 and 2.3 are special cases of more general results. (For example: [3], IV.5 and IV.6; [8], V.11.2.)

PROPOSITION 2.2. Let $\mathbf{G} = (G, \circ)$ be a groupoid and $\mathbf{R} = (R, *)$ be a free groupoid in $\mathcal{V}_{m,n}$ with the basis G . Let \approx be the least congruence on \mathbf{R} with the property

$$a \circ b = c \Rightarrow a^m * b^n \approx c. \quad (2.2)$$

Then: $\mathbf{G} \in \mathcal{V}_{m,n}^\square$ iff the following condition is satisfied:

$$(\forall a, b \in G)[a \approx b \Rightarrow a = b]. \quad \diamond$$

PROPOSITION 2.3. $\mathcal{V}_{m,n}^\square$ is a quasi-variety, i.e. there exists a system of axioms of $\mathcal{V}_{m,n}^\square$ each of which is a quasi-identity.³⁾ \diamond

³⁾ *Pr.* 2.2 is almost obvious and *Pr.* 2.3 is a corollary of it. Moreover, we can use *Pr.* 2.2 to obtain a convenient axiom system for $\mathcal{V}_{m,n}^\square$. Such a procedure is exposed in [2], where it is found an axiom system of quasi-identities for the quasi-variety of algebras $\mathbf{A} = (A, \Omega)$ which can be embedded in semigroups $\mathbf{S} = (S, \cdot)$ in such a way that $f(a_1, \dots, a_n) = a_1 \dots a_n$, for each n -ary operator $f \in \Omega_n$ ($n \geq 2$).

PROPOSITION 2.4. *The quasi-identity*

$$x \circ x = y \circ y \Rightarrow x \circ z = y \circ z \quad (2.3)$$

is true in each groupoid $\mathbf{G} = (G, \circ) \in \mathcal{V}_{m,n}^{\square}$. \diamond

PROPOSITION 2.5. $\mathcal{V}_{m,n}^{\square}$ is a proper subclass of the class of groupoids.

Proof. Let $\mathbf{G} = (\{a, b\}, \cdot)$ be a two-element groupoid such that $ba = b$, and $xy = a$ in every other case. Then (2.3) is not satisfied in \mathbf{G} . \diamond

Below we will establish some properties of the groupoid $\mathbf{R}^{\square} = (R, \square)$, assuming that $m > n \geq 2$. First recall that

$$u \square v = u^m * v^n, \quad (2.1')$$

for all $u, v \in R$.

In the Pr. 2.6–2.11 we assume that $m > n \geq 2$. They are corollaries of the definitions of \mathbf{R} and \mathbf{R}^{\square} , and the injectivity of \mathbf{F} .

PROPOSITION 2.6. x^n is a prime in R^{\square} , for each $x \in \mathbf{R}$. \diamond

PROPOSITION 2.7. If $(m, n) \notin \{(3, 2), (4, 3)\}$, then \mathbf{R}^{\square} is injective. \diamond

PROPOSITION 2.8. Let $u, v, \gamma, \delta \in R$ and $(u, v) \neq (\gamma, \delta)$.

1) If $(m, n) = (3, 2)$, then:

$u \square v = \gamma \square \delta$ iff $\{(u, v), (\gamma, \delta)\} = \{(y^2, y), (y, y)\}$, for some $y \in R$.

2) If $(m, n) = (4, 3)$, then: $u \square v = \gamma \square \delta$ iff

$$\{(u, v), (\gamma, \delta)\} = \{(z^{(r+1)}, z^{(r)}), (z^{(s+1)}, z^{(s)})\},$$

for some $z \in R$ and $0 \leq r < s$. \diamond

PROPOSITION 2.9. The subgroupoid \mathbf{Q} of \mathbf{R}^{\square} generated by the basis B of \mathbf{R} is injective. ⁴⁾

Proof. If $(m, n) \in \{(3, 2), (4, 4)\}$, the assertion is a corollary from Pr. 2.6 and Pr. 2.8; in the case $m > n + 1$ or $m \geq 5$ we can apply Pr. 2.7. \diamond

PROPOSITION 2.10. Only trivial identities hold in $\mathcal{V}_{m,n}^{\square}$. \diamond

Finally:

PROPOSITION 2.11. $\mathcal{V}_{m,n}^{\square}$ is not a variety.

⁴⁾ We note (see, for example: [3], IV.4.4) that \mathbf{Q} is free in $\mathcal{V}_{m,n}^{\square}$ with basis B .

Proof. If $\mathcal{V}_{m,n}^\square$ were a variety, then by *Pr.* 2.10, it would be defined by a trivial identity, for example $x = x$. This would imply that $\mathcal{V}_{m,n}^\square$ is the class of all groupoids, which contradicts *Pr.* 2.5. \diamond

Thus the proof of *Th.* 2 is completed.

The following two propositions are corollaries of *Pr.* 1.6–1.7 and the definitions of $\mathcal{V}_{2,n}$ and $\mathcal{V}_{2,n}^\square$. We see from them that the condition $m \geq 3$ is essential for *Th.* 2.

PROPOSITION 2.12. *For every $n \geq 1$, $\mathcal{V}_{1,n}^\square$ is the variety of left-zero groupoids.* \diamond

PROPOSITION 2.13. *For every $n \geq 1$, $\mathcal{V}_{2,n}^\square = \mathcal{V}_{2,n}$.* \diamond

3. $\mathcal{V}_{m,n}$ -reduced sets

Assume that \mathcal{V} is a (non-trivial) variety of groupoids, and $\mathbf{F} = (F, \cdot)$ a free groupoid with the basis B . Let $\approx_{\mathcal{V}}$ (further on: \approx) be the least congruence on \mathbf{F} such that $\mathbf{F}/\approx \in \mathcal{V}$. If $u \in F$, then we denote by u/\approx the \approx -class containing u . We say that a subset S of F is \mathcal{V} -reduced iff the mapping $u \mapsto u/\approx$ is a bijection from S onto F/\approx . Thus:

PROPOSITION 3.1. *Let S be a \mathcal{V} -reduced set of F and the operation \bullet is defined on S as follows:*

$$u, v, w \in S \Rightarrow (u \bullet v = w \Leftrightarrow uv \approx w). \tag{3.1}$$

Then $u \mapsto u/\approx$ is an isomorphism from $\mathbf{S} = (S, \bullet)$ onto \mathbf{F}/\approx , and \mathbf{S} is free in \mathcal{V} with the basis B . \diamond

PROPOSITION 3.2. *$R_{m,n}$ is $\mathcal{V}_{m,n}$ -reduced set iff: $m = 1$ or $m > n \geq 2$.*

Proof. If $m > n \geq 2$ or $m = 1$, then from *Th.* 1 and *Pr.* 1.6–1.7 follows that $R_{m,n}$ is a $\mathcal{V}_{m,n}$ -reduced set. Namely, the rewriting system (on \mathbf{F}) induced by elementary transformations: $u^m v^n \rightarrow \underline{uvm}$ is a terminating Church–Rose system ([5], 2.9), and $R_{m,n}$ consists of the normal forms in this system.

Let $m \geq 2$, $n = 1$ and $a \in B$.

If $m = 2$, then:

$$\begin{aligned} (a^2)^2 a &\rightarrow a^2 a \rightarrow aa = a^2 \in R_{2,1}, \\ (a^2)^2 a &= (a^2 a^2) a \rightarrow (aa^2) a \in R_{2,1}. \end{aligned}$$

If $m \geq 3$, then:

$$\begin{aligned} (a^m)^m a &\rightarrow \underline{a^m a m} = a^m a \underline{a^m a m - 2} \rightarrow a^m a^m \underline{a a^m m - 3} \rightarrow \\ &\rightarrow \underline{a a^m m a a^m m - 3} \in R_{m,1}, \end{aligned}$$

$$(a^m)^m a = a^m a^m \underline{a^m a^m m-2 a} \rightarrow \underline{a a^m m a^m a^m m-2 a} \in R_{m,1}.$$

If $m = n \geq 2$, then:

$$(a^n)^{n+1} = (a^n)^n a^n \rightarrow \underline{a^n a^n} \in R_{n,n},$$

$$\begin{aligned} (a^n)^{n+1} &= a^n a^n \underline{a^n a^n n-1} \rightarrow a^n \underline{a^n a^n n-1} = \\ &= (a^n)^n \rightarrow (a^n)^{n-1} \rightarrow \dots \rightarrow a^n \in R_{n,n}. \end{aligned}$$

Finally, if $2 \leq m < n$, then:

$$(a^n)^m (a^n)^n \rightarrow \underline{a^n a^n m} = (a^n)^m \in R_{m,n},$$

$$\begin{aligned} (a^n)^m (a^n)^n &= (a^n)^m ((a^n)^m a^n) \underline{a^n a^n n-m-1} \rightarrow \\ &\rightarrow (a^n)^m (\underline{a^n a^n m a^n a^n n-m-1}) \in R_{m,n}. \end{aligned}$$

Therefore, if $m > n = 1$ or $2 \leq m \leq n$, then there exist $u, v \in R_{m,n}$ such that $u \neq v$, $u \approx v$, i.e. $R_{m,n}$ is not $\mathcal{V}_{m,n}$ -reduced. \diamond

From Pr. 3.2 follows that the definition of $R_{m,n}$ is "unsuccessful" if $m > n = 1$ or $2 \leq m \leq n$.

References

- [1] R. H. Bruck: *A Survey of Binary Systems*, Berlin-Göttingen-Heidelberg, 1956
- [2] N. Celakoski: *On Semigroup Associatives*, Macedonian Academy of Sciences and Arts, Contributions IX 2 (1977), 5-19.
- [3] R. M. Cohn: *Universal Algebra*, Harpers Series in Modern Mathematics, 1965.
- [4] Ć Čupona, N. Celakoski: *Polyadic Subsemigroups of Semigroups*, Algebraic Conference Skopje 1980, 131-152.
- [5] O. G. Kharalampovich, M. V. Sapir: *Algorithmic Problems in Varieties*, International Journal of Algebra and Computation, Vol. 5 (1995), 379-602.
- [6] S. Markovski: *On quasivarieties of generalized subalgebras*, Algebraic Conference Skopje 1980, 125-129.
- [7] M. D. Prešić, S. B. Prešić: *On the Embedding of Ω -algebras in Groups*, Publ. de l'Inst. Math. Nouv. sèrie, tome 21(35), 1977, 169-174.
- [8] А. И. Мальцев: *Алгебраические системы*, Москва 1970.

СЛОБОДНИ ОБЈЕКТИ ВО НЕКОИ МНОГУОБРАЗИЈА ГРУПОИДИ

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Резиме

Во работава се дава каноничен опис на слободните објекти во многуобразието групоиди $x^m y^n = x y x \dots$, каде што $m > n \geq 2$, а на десната страна се појавуваат m фактори, по ред: x, y, x, y, \dots

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