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PRIMITVE n-IDENTITIES

Abstract. A primitive n-identity is a formula of the form (0.1) in the class of n-groupoids (i.e. algebras with one n-ary operation). Every such a formula is interpreted as a transformation of the set $\underline{2n}$ = = {1,2,...,2n}, and as a partition of $\underline{2n}$, as well. Completeness theorems are obtained for the both such interpretations. Some results about the lattices of primitive varieties (defined by primitive n-identities) are also mentioned.

0. Preliminaries. The subject of this paper are the primitive n-identities, i.e. the identities in the class of n-groupoids of the following form

$$x_{\varphi(1)}x_{\varphi(2)}\dots x_{\varphi(n)} = x_{\varphi(n+1)}x_{\varphi(n+2)}\dots x_{\varphi(2n)} \qquad (0.1)$$

where $\varphi: \underline{2n} \to \underline{2n}$ is a transformation of the set $\underline{2n} = \{1,2,3,\dots,2n\}$[1]. (Note that, in the case n = 1, $x_1 = x_2$ means that $f(x_1) = f(x_2)$, where f is a unary operator.) From now on, an "identity" will mean a "primitive n-identity", and we will assume that n is a given positive integer.

Let $\mathcal{T} = \{\varphi \mid \varphi: \underline{2n} \to \underline{2n}\}$ ($= \mathcal{T}_{2n}$) be the set of all transformations of $\underline{2n}$. The identities can be considered as elements of $\mathcal{T}$ in the following way. Let $\underline{G} = (G;f)$ be an n-groupoid where G is a non-empty set and $f: G^n \to G$ a mapping. We say that φ is an identity in $\underline{G}$ iff for each mapping $\eta: \underline{2n} \to G$

$$f((\eta\cdot\varphi)(1),\dots,(\eta\cdot\varphi)(n)) = f((\eta\cdot\varphi)(n+1),\dots,(\eta\cdot\varphi)(2n)).$$

(In the case n = 1 we have that $x_1 = x_2$ is an identity in (G;f) iff $f(a_1) = f(a_2)$ for each $a_1, a_2 \in G$.)

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[1] Certainly, we can allow φ to be a transformation of the set of positive integers, but it can be easily seen that every such an identity is equivalent to an identity

$$x_{\psi(1)}x_{\psi(2)}\dots x_{\psi(n)} = x_{\psi(n+1)}x_{\psi(n+2)}\dots x_{\psi(2n)},$$

where ψ is a transformation of $\underline{2n}$.

If $\mathscr{C}$ is a class of n-groupoids, we say that $\varphi \in \mathscr{T}$ is an identity in $\mathscr{C}$ if φ is an identity in each $(G;f) \in \mathscr{C}$.

Let $\Sigma \subseteq \mathscr{T}$. If $\mathscr{C}$ is the class of all n-groupoids in which every $\varphi \in \Sigma$ is an identity, we say that $\mathscr{C}$ is the primitive variety of n-groupoids determined by the set of identities Σ, and we write $\mathscr{C} =$ $=$ Var Σ. Var Σ will be refered simply as a variety.

Since the set $\mathscr{T}$ has $(2n)^{2n}$ elements, it follows that there are only finitely many varieties.

Next, let $\varphi, \psi \in \mathscr{T}$. We say that ψ is a consequence of φ, and write $\varphi \models \psi$, if ψ is an identity in each n-groupoid $\underline{G}$ in which φ is an identity.

Similarly, if Σ, $\Lambda \subseteq \mathscr{T}$, we say that Λ is a consequence of Σ, and write $\Sigma \models \Lambda$, if each $\psi \in \Lambda$ is an identity in each n-groupoid $\underline{G}$ in which every $\varphi \in \Sigma$ is an identity. If $\Sigma = \emptyset$ and $\varphi \in \mathscr{T}$, instead of $\emptyset \models \varphi$ we write simply $\models \varphi$. It is clear that

$$\text{Var } \Sigma \subseteq \text{Var } \Lambda \quad \text{iff} \quad \Sigma \models \Lambda.$$

Hence, the notion of a consequence corresponds to the notion of a subvariety.

For Σ, $\Lambda \subseteq \mathscr{T}$, we say that they are equivalent, and write $\Sigma \models\!\!\!\dashv \Lambda$, iff $\Sigma \models \Lambda$ and $\Lambda \models \Sigma$, which means that Var $\Sigma =$ Var Λ.

It is easy to check that $\models$ is a reflexive and transitive relation on the boolean $\mathscr{B}(\mathscr{T}) = \{\Sigma \mid \Sigma \subseteq \mathscr{T}\}$. Thus, $\models\!\!\!\dashv$ is an equivalence relation on $\mathscr{B}(\mathscr{T})$.

A set of identities $\Sigma \subseteq \mathscr{T}$ is said to be complete if Σ contains all identities which are consequences of Σ, i.e. if the following implication is true:

$$\Sigma \models \varphi \quad \Rightarrow \quad \varphi \in \Sigma.$$

The main result of this paper is the description of the complete subsets of identities of $\mathscr{T}$ as ideals in an algebra over $\mathscr{T}$ (Theorem 3.1), and as its corollary we obtain the lattice of all primitive varieties of n-groupoids.

1. The algebra $\underline{\mathcal{T}} = (\mathcal{T};\varepsilon,{}^s,*,\cdot)$. We define an algebra over $\mathcal{T}$ with a nulary operation ε, a unary operation ^{s}, and two binary operations $*$ and $\cdot$, where $\cdot$ is the usual composition of mappings.

Let $\mathcal{T}' = \{(\alpha,\beta) \mid \alpha,\beta\colon \underline{n} \to \underline{2n}\}$, and for each $\varphi \in \mathcal{T}$ define two mappings φ_L, φ_R: $\underline{n} \to \underline{2n}$ by

$$\varphi_L(i) = \varphi(i),\ \varphi_R(i) = \varphi(i+n), \text{ for each } i \in \underline{n}.$$

1.1. The mapping $h\colon \mathcal{T} \to \mathcal{T}'$, defined by $h(\varphi) = (\varphi_L,\varphi_R)$, is a bijection.

Proof. If $(\alpha,\beta) \in \mathcal{T}'$, then define a $\varphi \in \mathcal{T}$ by $\varphi(i) = \alpha(i)$ and $\varphi(i+n) = \beta(i)$ for each $i \in \underline{n}$. Then $\varphi_L = \alpha$, $\varphi_R = \beta$. ■

From now on we identify the elements $\varphi \in \mathcal{T}$ and $(\varphi_L,\varphi_R) \in \mathcal{T}'$. Thus, $(\alpha,\beta)_L = \alpha$ and $(\alpha,\beta)_R = \beta$ for each $(\alpha,\beta) \in \mathcal{T}'$, i.e. $(\alpha,\beta)(i) = \alpha(i)$ and $(\alpha,\beta)(i+n) = \beta(i)$ for each $i \in \underline{n}$. It is clear that

1.2. If $\varphi,(\alpha,\beta) \in \mathcal{T}$, then $\varphi\cdot(\alpha,\beta) = (\varphi\cdot\alpha,\varphi\cdot\beta)$. ■

Now, let $\varepsilon \in \mathcal{T}$ be defined by $\varepsilon_L(i) = \varepsilon_R(i) = i$, and for each $\varphi \in \mathcal{T}$ define $\varphi^s \in \mathcal{T}$ by $\varphi^s = (\varphi_R,\varphi_L)$, i.e. $(\varphi^s)_L = \varphi_R$, $(\varphi^s)_R = \varphi_L$.

Finally, define a binary operation $*$ on $\mathcal{T}$ by:

$$(\alpha,\beta)*(\gamma,\delta) = \begin{cases} (\alpha,\delta) & \text{if } \beta = \gamma \\ \varepsilon & \text{if } \beta \neq \gamma \end{cases}$$

for all $(\alpha,\beta),(\gamma,\delta) \in \mathcal{T}$.

In such a way an algebra $\underline{\mathcal{T}} = (\mathcal{T};\varepsilon,{}^s,*,\cdot)$ has been defined.

2. Some properties of $\models$. Here we give some facts about $\models$.

2.1. Let $\varphi,\psi \in \mathcal{T}$. Then:

(i) $\varphi \models \psi\cdot\varphi$.

(ii) If $\ker\varphi \subseteq \ker\psi$, then $\varphi \models \psi$.

Proof. (i) If φ is an identity in an n-groupoid $(G;f)$, then $f(\eta\cdot\varphi(1),\dots,\eta\cdot\varphi(n)) = f(\eta\cdot\varphi(n+1),\dots,\eta\cdot\varphi(2n))$ for each mapping $\eta\colon \underline{2n} \to G$. Now we can replace η by $\eta\cdot\psi$, obtaining

$$f(\eta\cdot\psi\cdot\varphi(1),\dots,\eta\cdot\psi\cdot\varphi(n)) = f(\eta\cdot\psi\cdot\varphi(n+1),\dots,\eta\cdot\psi\cdot\varphi(2n))$$

for each mapping $\eta\colon \underline{2n} \to G$, i.e. $\psi\cdot\varphi$ is an identity in $(G;f)$ too.

(ii) By (i), it is enough to show that if $\ker\varphi \subseteq \ker\psi$, then there is some $\xi \in \mathcal{T}$ such that $\xi\cdot\varphi = \psi$. Such a ξ can be defined in this way: if $j \in \varphi^{-1}(i)$, then we put $\xi(i) = \psi(j)$, and $\xi(i)$ is arbitrary otherwise. ■

The converse of 2.1. is not true in general, as shows the next

Example 2.2. Let n = 2 and consider the transformations $\varphi, \psi \in \mathcal{T}$ defined by

$$\varphi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 4 \end{bmatrix}, \qquad \psi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{bmatrix}.$$

Then $x_1x_1 = x_3x_4$, $x_1x_2 = x_3x_3$ are corresponding "usual" forms of the identites φ, ψ. Clearly, $\varphi \models \psi$ but $ker\ \varphi$ is not a subset of $ker\ \psi$. ■

The following statements are clear:

2.3. If $\varphi, \psi \in \mathcal{T}$ and $ker\ \varphi = ker\ \psi$, then $\varphi \mathrel{\vert\!\!\!\!\dashv\!\!\vdash} \psi$. ■

2.4. If $\varphi \in \mathcal{T}$ and $\varphi_L = \varphi_R$, then $\models \varphi$. ■

2.5. $\varphi \mathrel{\vert\!\!\!\!\dashv\!\!\vdash} \varphi^s$ for each $\varphi \in \mathcal{T}$. ■

2.6. If $\varphi, \psi \in \mathcal{T}$, then $\varphi, \psi \models \varphi * \psi$. ■

3. The completness theorem. A nonempty subset $\mathcal{J} \subseteq \mathcal{T}$ is said to be an ideal in $\underline{\mathcal{T}}$ if $\mathcal{J}$ is a subalgebra of $\underline{\mathcal{T}}$ such that $\varphi \cdot \psi \in \mathcal{J}$ for each $\varphi \in \mathcal{T}$, $\psi \in \mathcal{J}$, i.e. if

1) $\varepsilon \in \mathcal{J}$,

2) $\varphi \in \mathcal{J} \Rightarrow \varphi^s \in \mathcal{J}$,

3) $\varphi, \psi \in \mathcal{J} \Rightarrow \varphi * \psi \in \mathcal{J}$, and

4) $\varphi \in \mathcal{T}$, $\psi \in \mathcal{J} \Rightarrow \varphi \cdot \psi \in \mathcal{J}$,

for each $\varphi, \psi \in \mathcal{T}$.

Here we characterize the complete subsets in $\mathcal{T}$ by using the ideals in $\underline{\mathcal{T}}$.

Theorem 3.1. A subset Σ of $\mathcal{T}$ is a complete set of identities iff Σ is an ideal in $\underline{\mathcal{T}}$.

Proof. Let $\Sigma \subseteq \mathcal{T}$ be a complete set of identities. Then 1) - 4) are satisfied by 2.1, 2.4, 2.5 and 2.6.

Conversely, let Σ be an ideal in $\mathcal{T}$. We will show that if $\varphi \in \mathcal{T} \setminus \Sigma$ then φ is not a consequence of Σ. So, let $(\varphi_L, \varphi_R) \notin \Sigma$, and put

$$\Lambda = \{\alpha \mid \alpha : \underline{n} \longrightarrow \underline{2n},\ (\varphi_L, \alpha) \in \Sigma\}.$$

Note that $(\varphi_L, \varphi_L) = \varphi \cdot \varepsilon \in \Sigma$ and $(\varphi_L, \varphi_R) \notin \Sigma$ which implies that $\varphi_L \in \Lambda$ and the set $\mathcal{T} \setminus \Lambda$ is nonempty.

Now we take $G = \underline{2n}$ and define an n-ary operation $f: G^n \longrightarrow G$ as follows. Let $p_1, \ldots, p_n \in G = \underline{2n}$. Then $f(p_1, \ldots, p_n) = 1$ iff there is an $\alpha \in \Lambda$ such that $\alpha(1) = p_1, \ldots, \alpha(n) = p_n$, and $f(p_1, \ldots, p_n) = 2$ otherwise.

Then φ is not an identity in $(G;f)$, since for the identity mapping η on $\underline{2n}$ (i.e. $\eta(i) = i$ for each $i \in \underline{2n}$) we have.

$$f(\eta\cdot\varphi(1),\dots,\eta\cdot\varphi(n)) = f(\varphi_L(1),\dots,\varphi_L(n)) = 1,$$
$$f(\eta\cdot\varphi(n+1),\dots,\eta\cdot\varphi(2n)) = f(\varphi_R(1),\dots,\varphi_R(n)) = 2.$$

To complete the proof, it is enough to show that $(G;f) \in \mathrm{Var}\,\Sigma$.

Let $\psi = (\psi_L,\psi_R) \in \Sigma$, and let $\eta:\underline{2n} \longrightarrow G = \underline{2n}$ be any mapping. Then $\eta \in \mathcal{T}$, and $\eta\cdot\psi = (\eta\cdot\psi_L,\eta\cdot\psi_R) \in \Sigma$. We will show that

$$\eta\cdot\psi_L \in \Lambda \iff \eta\cdot\psi_R \in \Lambda.$$

Namely, if $\eta\cdot\psi_L \in \Lambda$, i.e. $(\varphi_L,\eta\cdot\psi_L) \in \Sigma$, then $(\varphi_L,\eta\cdot\psi_R) = (\varphi_L,\eta\cdot\psi_L) * * (\eta\cdot\psi_L,\eta\cdot\psi_R) \in \Sigma$, and so $\eta\cdot\psi_R \in \Lambda$; if $\eta\cdot\psi_R \in \Lambda$, i.e. $(\varphi_L,\eta\cdot\psi_R) \in \Sigma$, then $(\varphi_L,\eta\cdot\psi_L) = (\varphi_L,\eta\cdot\psi_R) * (\eta\cdot\psi_L,\eta\cdot\psi_R)^s \in \Sigma$, and so $\eta\cdot\psi_L \in \Lambda$.

Finally we have

$$f(\eta\cdot\psi(1),\dots,\eta\cdot\psi(n)) = 1 \iff f(\eta\cdot\psi_L(1),\dots,\eta\cdot\psi_L(n)) = 1$$
$$\iff \eta\cdot\psi_L \in \Lambda \iff \eta\cdot\psi_R \in \Lambda \iff f(\eta\cdot\psi_R(1),\dots,\eta\cdot\psi_R(n)) = 1$$
$$\iff f(\eta\cdot\psi(n+1),\dots,\eta\cdot\psi(2n)) = 1,$$

which means that ψ is an identity in $(G;f)$. ■

It is clear that the set of all ideals in $\underline{\mathcal{T}}$ is a lattice, where the meet is the usual set theoretical intersection, and the joint of two ideals Σ_1 and Σ_2 is the least ideal Σ containing them. Thus we have:

Corollary 3.2. The lattice of (primitive) varieties of n-groupoids is antiisomorphic to the lattice of ideals in the algebra $\underline{\mathcal{T}}$.

■

4. Partitions as identities. A partiton of a set A is called any subset π of the boolean $\mathcal{B}(A) = \{X \mid X \subseteq A\}$ having the following three properties:

(i) $X, Y \in \pi,\ X \cap Y \neq \emptyset \Rightarrow X = Y$;

(ii) $A = \cup\,(X \mid X \in \pi)$;

(iii) $\emptyset \in \pi$.[2)]

Here we will show how partitions of the set $\underline{2n}$ can be considered as (primitive n-) identities.

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2) The condition (iii) is not essential.

As we already noted in 2.3, if $\varphi, \psi \in \mathcal{T}$ and $ker\ \varphi = ker\ \psi$, then $\varphi \models\!\!\dashv \psi$. Now, if π is a partition of $\underline{2n}$, then π represent a set of eqivalent identities, namely that is the subset $\{\varphi \mid \underline{2n} /_{ker \varphi} = \pi\}$ of $\mathcal{T}$. So, from now on, for any partition π of $\underline{2n}$ we will say that it is an identity. Denote by $\mathcal{P}$ (= $\mathcal{P}_{2n}$) the set of all partitions of $\underline{2n}$, and one can show that $|\mathcal{P}_{2n}| < |\mathcal{T}_{2n}|^{1/2} = (2n)^n$. In such a way the number of identities (partitions) to deal with is considerably shortened. Example 2.2 shows that different partitions can represent equivalent identities, and so we have to give a considerable characterization of sets of identities (as partitions) too.

Further on we will use the following notations. If $\varphi \in \mathcal{T}$, then by $\varphi^{\wedge}$ (= $\underline{2n} /_{ker\varphi}$) is denoted the member of $\mathcal{P}$ determined by $ker\ \varphi$, and if $\Sigma \subseteq \mathcal{T}$, then $\Sigma^{\wedge} = \{\varphi^{\wedge} \mid \varphi \in \Sigma\}$. If $\pi \in \mathcal{P}$, then we put $\pi^{\vee} = \{\varphi \in \mathcal{T} \mid \varphi^{\wedge} = \pi\}$, and if $\Lambda \subseteq \mathcal{P}$, then $\Lambda^{\vee} = \{\varphi \in \mathcal{T} \mid \varphi^{\wedge} \in \Lambda\} = \cup\, (\pi^{\vee} \mid \pi \in \Lambda)$.

4.1. For any subsets $\Lambda \subseteq \mathcal{P}$ and $\Sigma \subseteq \mathcal{T}$, we have

$\Lambda^{\vee\wedge} = \Lambda, \quad \Sigma \subseteq \Sigma^{\wedge\vee}$. ∎

Let $\pi, \tau \in \mathcal{P}$. Then we say that τ is a consequence of π and write $\pi \models \tau$ iff $\pi^{\vee} \models \tau^{\vee}$ (i.e. iff $\pi^{\vee} \models \psi$ for every $\psi \in \mathcal{T}$ such that $\psi^{\wedge} = \tau$). In the same manner, if $\Lambda \subseteq \mathcal{P}$, $\pi \in \mathcal{P}$, then $\Lambda \models \pi$ iff $\Lambda^{\vee} \models \pi^{\vee}$. As before, we write $\pi \models\!\!\dashv \tau$ iff $\pi \models \tau$ and $\tau \models \pi$, and we say that π and τ are equivalent.

A subset $\Lambda \subseteq \mathcal{P}$ is said to be complete in $\mathcal{P}$ iff

$\Lambda \models \pi \Rightarrow \pi \in \Lambda$, for each $\pi \in \mathcal{P}$.

4.2. Let $\Lambda \subseteq \mathcal{P}$. Then Λ is complete in $\mathcal{P}$ iff $\Lambda^{\vee}$ is complete in $\mathcal{T}$.

Proof. Let Λ be complete, and suppose $\Lambda^{\vee} \models \varphi$ for some $\varphi \in \mathcal{T}$. If $ker\ \psi = ker\ \varphi$ for some $\psi \in \mathcal{T}$, then $\varphi \models\!\!\dashv \psi$, and we have $\Lambda^{\vee} \models \psi$ as well. This means that $\Lambda^{\vee\wedge} = \Lambda \models \varphi^{\wedge}$, i.e. $\varphi^{\wedge} \in \Lambda$, which implies $\varphi \in \Lambda^{\vee}$.

Let $\Lambda^{\vee}$ be complete in $\mathcal{T}$, and $\Lambda \models \pi$ for some $\pi \in \mathcal{P}$. Then $\Lambda^{\vee} \models \varphi$ for any $\varphi \in \mathcal{T}$ such that $\varphi^{\wedge} = \pi$. Thus we have $\varphi \in \Lambda^{\vee}$, i.e. $\varphi^{\wedge} = \pi \in \Lambda^{\wedge\vee} = \Lambda$. ∎

The complete subsets of $\mathcal{P}$ can be characterized in a manner similar to that for complete subsets of $\mathcal{T}$. We first give some necessary definitions.

Let $\pi \in \mathcal{P}$. Then by π^s we denote the partition of $\mathcal{P}$ induced by the mapping $^s: i \mapsto n+i,\ \ i+n \mapsto i,$ for each $i \in \underline{n}$. This means that

$$A \in \pi \quad \text{iff} \quad A^s = \{i^s \mid i \in A\} \in \pi^s.$$

Next, by $\pi_L = \pi|_{\underline{n}}$ we denote the restriction of π on $\underline{n}$, and we put $\pi_R = (\pi^s)_L$. Clearly, π_L and π_R are partitions of $\underline{n}$.

If $\tau, \pi \in \mathcal{P}$ and $\tau = \pi^s$, then we say that τ is simmetric to π.

4.3. (i) $\pi \in \mathcal{P} \Rightarrow (\pi^s)^s = \pi,\ \pi^s \models\!\!\mid \pi$.

(ii) $\varphi \in \mathcal{T} \Rightarrow (\varphi^s)^\wedge = (\varphi^\wedge)^s$. ∎

Let $\pi \in \mathcal{P}$ and $i \in \underline{2n}$, $j \in \underline{n}$. Then by $\pi(i)$ (by $\pi_L(j), \pi_R(j)$) we denote the member of π (of π_L, π_R) containing the element i (element j). It is clear that the relation $\le$ defined on $\mathcal{P}$ by:

$$\sigma, \tau \in \mathcal{P} \Rightarrow (\sigma \le \tau \iff (\forall\, i \in \underline{2n})(\sigma(i) \subseteq \tau(i))$$

is an ordering relation on $\mathcal{P}$.

4.4. (i) $\sigma, \tau \in \mathcal{P} \Rightarrow (\sigma \le \tau \Rightarrow \sigma \models \tau)$.

(ii) $\varphi, \psi \in \mathcal{T} \Rightarrow \varphi^\wedge \le (\psi \cdot \varphi)^\wedge$. ∎

Let $\sigma, \tau \in \mathcal{P}$ be such that $\sigma_R = \tau_L$. Then by $\sigma * \tau$ we denote the smallest partition $\pi \in \mathcal{P}$ having the following two properties:

1) $\underline{n} \cap \sigma(i) \subseteq \pi(i)$, $(\underline{2n} \setminus \underline{n}) \cap \tau(i+n) \subseteq \pi(i+n)$ for each $i \in \underline{n}$.

2) $\sigma(i) = \sigma(k+n)$, $\tau(k) = \tau(j+n) \Rightarrow \pi(i) = \pi(j+n)$ for each $i, k, j \in \underline{n}$.

If $\sigma, \tau \in \mathcal{P}$ and $\sigma_R \ne \tau_L$ then we put $\sigma * \tau = \rho$, where by ρ we denote the partition $\{\{i, i+n\} \mid i \in \underline{n}\} = \varepsilon^\wedge$.

4.5. $\varphi, \psi \in \mathcal{T} \Rightarrow \varphi^\wedge * \psi^\wedge \le (\varphi * \psi)^\wedge$. ∎

In such a way an algebra $\underline{\mathcal{P}} = (\mathcal{P};\ \rho,\ ^s,\ *)$ with a carrier $\mathcal{P}$, a nullary operation ρ, a unary operation ^{s} and a binary operation $*$ is obtained. A subset $\Lambda \subseteq \mathcal{P}$ is said to be a filter in $\underline{\mathcal{P}}$ iff Λ is a subalgebra of $\underline{\mathcal{P}}$ and

$$\sigma \in \Lambda,\ \tau \in \mathcal{P},\ \sigma \le \tau \Rightarrow \tau \in \Lambda.$$

4.6. Let $\Lambda \subseteq \mathcal{P}$. Then Λ is a filter in $\underline{\mathcal{P}}$ iff $\Lambda^\vee$ is an ideal in $\underline{\mathcal{T}}$.

Proof. Let Λ be a filter in $\underline{\mathcal{P}}$. Then $\varepsilon \in \Lambda^\vee$, since $\varepsilon^\wedge = \rho \in \Lambda$. If $\varphi, \psi \in \Lambda^\vee$, then $\varphi^\wedge, \psi^\wedge \in \Lambda$, which implies $(\varphi^\wedge)^s \in \Lambda$ and $\varphi^\wedge * \psi^\wedge \in \Lambda$. By 4.3(ii) we have $\varphi^s \in \Lambda^\vee$, since $(\varphi^s)^\wedge \in \Lambda$. On the other hand, by 4.5 we have $\varphi^\wedge * \psi^\wedge \le (\varphi * \psi)^\wedge$, and so $(\varphi * \psi)^\wedge \in \Lambda$, i.e. $\varphi * \psi \in \Lambda^\vee$. If $\varphi \in \Lambda^\vee$ and $\xi \in \mathcal{T}$, then by 4.4(ii) we have $\varphi^\wedge \le (\xi \cdot \varphi)^\wedge$ and so $(\xi \cdot \varphi)^\wedge \in \Lambda$, i.e. $\xi \cdot \varphi \in \Lambda^\vee$.

Let $\Lambda^{\vee}$ be an ideal in $\underline{\mathcal{T}}$. Then $\rho = \varepsilon^{\wedge} \in \Lambda^{\vee\wedge} = \Lambda$. Take $\sigma \in \Lambda$. Then $\sigma = \varphi^{\wedge}$ for some $\varphi \in \Lambda^{\vee}$, and $\varphi^{s} \in \Lambda^{\vee}$ implies $\sigma^{s} = (\varphi^{\wedge})^{s} = (\varphi^{s})^{\wedge} \in \Lambda$. Let $\sigma, \tau \in \Lambda$ and $\sigma_R = \tau_L$. Define inductively the elements $\varphi, \psi \in \mathcal{T}$ as follows.

First, for each $i \in \underline{2n}$, we define φ by

$$\varphi(i) = \begin{cases} \varphi(j), & \text{if } i \in \sigma(j) \text{ for some } j < i, \\ \min \underline{2n} \setminus \{\varphi(j) \mid j = 1,2,\ldots,i-1\}, & \text{otherwise} \end{cases}$$

and, after that, for each $i \in \underline{n}$, we define $\psi \in \mathcal{T}$ by

$\psi(i) = \varphi(i+n)$,

$$\psi(i+n) = \begin{cases} \psi(j), & \text{if } i+n \in \tau(j) \text{ for some } j < i+n \\ \max \underline{2n} \setminus \{\psi(j) \mid j = 1,2,\ldots,i+n-1\}, & \text{otherwise} \end{cases}$$

In such a way we have that $\varphi^{\wedge} = \sigma$, $\psi^{\wedge} = \tau$ and so $\varphi, \psi \in \Lambda^{\vee}$. We note that $|\varphi(\underline{2n})| = |\sigma|$, $|\psi(\underline{2n})| = |\tau|$ and $\varphi_R = \psi_L$. By the definition of φ and ψ we have $\sigma * \tau \le (\varphi * \psi)^{\wedge}$, and we will define a $\psi' \in \mathcal{T}$ such that $\psi' \in \Lambda^{\vee}$ and $\sigma * \tau = (\varphi * \psi')^{\wedge}$, which will implies $\sigma * \tau \in \Lambda$. Namely, for each $i \in \underline{n}$, we define a $\xi \in \mathcal{T}$ as follows:

$\xi(\psi(i)) = \psi(i)$,

$$\xi(\psi(i+n)) = \begin{cases} \psi(i+n), & \text{if } \psi(i+n) = \psi(j) \text{ for some } j < i+n \\ \max \ \underline{2n} \setminus (\varphi(\underline{n}) \cup \{\xi(\psi(j)) \mid n < j < i+n\}), & \text{otherwise} \end{cases}$$

$\xi(j)$ is arbitrary if $j \notin \psi(\underline{2n})$.

Then it is enough to take $\psi' = \xi \cdot \psi$ for what we needed.

Let $\sigma \in \Lambda$, $\pi \in \mathcal{P}$, $\sigma \le \pi$ and $\sigma = \varphi^{\wedge}$ for some $\varphi \in \Lambda^{\vee}$. There is a $\xi \in \mathcal{T}$ such that $\pi = (\xi \cdot \varphi)^{\wedge}$ and, since $\xi \cdot \varphi \in \Lambda^{\vee}$, we have $\pi \in \Lambda$.

∎

Now, as a consequence of 4.2, 3.1 and 4.6 we have the following completness theorem:

Theorem 4.7. Let $\Lambda \subseteq \mathcal{P}$. Then Λ is complete in $\mathcal{P}$ iff Λ is a filter in $\underline{\mathcal{P}}$. ∎

Since the set of filters in $\underline{\mathcal{P}}$ is a lattice (with usually defined meet and joint) as a consequence we have:

Corollary 4.8. The lattice of (primitive) varieties of n-groupoids is antiisomorphic to the lattice of filters in the algebra $\underline{\mathcal{P}}$.

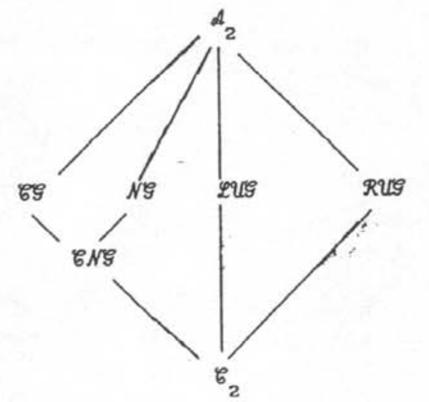
∎

5. A few remarks for the lattices of primitive varieties of n-groupoids, for n = 1,2,3. As we already mentioned in Corollaries 3.2 and 4.8 the lattice of primitive varieties of n-groupoids is antiisomorphic to the lattice of ideals in the algebra $\mathcal{I}$, i.e. to the lattice of filters in $\mathcal{P}$. For n = 1,2,3 we have $|\mathcal{I}_2| = 4$, $|\mathcal{I}_4| = 256$, $|\mathcal{I}_6| = 46656$, $|\mathcal{P}_2| = 2$, $|\mathcal{P}_4| = 15$, $|\mathcal{P}_6| = 203$, and the problem for construction of the sets of ideals and filters is an exponential one.

We note that, for each n, the class $\mathcal{A}_n$ of all n-groupoids and the class $\mathcal{C}_n$ of constant n-groupoids are primitive varieties, determined by the identities $x_1 \ldots x_n = x_1 \ldots x_n$, i.e. $x_1 \ldots x_n = x_{n+1} \ldots x_{2n}$. We say that these varieties are trivial ones. Clearly, if n = 1, we have only trivial varieties. For n = 2 there are 5 non-trivial varieties, defined by the identities:

$\mathcal{CG}$: $xy = yx$ - commutative groupoids
$\mathcal{NG}$: $x^2 = y^2$ - nil groupoids
$\mathcal{LUG}$: $xy = x^2$ - left-unar groupoids
$\mathcal{RUG}$: $xy = y^2$ - right-unar groupoids
$\mathcal{CNG}$: $xy = yx$, $x^2 = y^2$ - commutative nil groupoids

The Hasse diagram of the lattice of primitive variety of (binary) groupoids looks like



For n = 3 there are 287 varieties [3], and 49 of them can be defined by only one identity (which means that 47 of them are nontrivial). A list of these 49 identities (here x, y, z, u, v, w denote different variables), obtained by an exhausted checking, consists of:

$xyz = uvw$	$xyz = xuv$	$xyz = uyv$	$xyz = uvz$	$xyz = xyu$
$xyz = xuz$	$xyz = uyz$	$xyz = yxu$	$xyz = zux$	$xyz = uzy$
$xyz = yxz$	$xyz = zyx$	$xyz = xzy$	$xyz = yzx$	$x^2y = u^2v$
$x^2y = uvu$	$x^2y = uv^2$	$x^2y = uxu$	$x^2y = xu^2$	$x^2y = u^2y$
$x^2y = uyu$	$x^2y = yu^2$	$x^2y = ux^2$	$x^2y = xux$	$x^2y = x^2u$
$x^2y = uy^2$	$x^2y = yuy$	$x^2y = xyx$	$x^2y = yx^2$	$x^2y = y^2x$
$x^2y = yxy$	$x^2y = xy^2$	$xyx = uv^2$	$xyx = uvu$	$xyx = xu^2$
$xyx = uyu$	$xyx = yu^2$	$xyx = ux^2$	$xyx = xux$	$xyx = uy^2$
$xyx = yx^2$	$xyx = yxy$	$xyx = xy^2$	$xy^2 = uv^2$	$xy^2 = xu^2$
$xy^2 = uy^2$	$xy^2 = yx^2$	$x^3 = u^3$	$xyz = xyz$	

We mention that more results about the lattices of primitive varieties of n-groupoids can be find in the paper [3].

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