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TRANSITION TO CHAOS OF THE DUFFING OSCILLATOR MODIFIED

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Abstract

In this paper we apply Melnikov method for the Duffing oscillator modified. We change the last term of the Duffing oscillator. We will find the critical surface of the transition to chaos.

1. Introduction

In this section we recall the Melnikov method [3].
Let us note Y_ε the vector field:

$$Y_\varepsilon = \Big(\frac{\partial H}{\partial y_i}(x,y) + \varepsilon g_1(x,y,\theta)\,, -\frac{\partial H}{\partial x_i}(x,y) + \varepsilon g_2(x,y,\theta)\Big)\,.$$

We define the Melnikov function as: $M\colon \mathbf{R}\times S^1 \to \mathbf{R}$

$$M(s,\theta) = \int^{\infty} \langle \operatorname{grad} H(h(t-s)),\ g(h(t-s),\ \omega t+\theta)\rangle dt \quad \text{or}\,,$$

substituting $t \to t+s$ the Melnikov function is:

$$M(s,\theta) = \int_{-\infty}^{\infty} \langle \operatorname{grad} H(h(t-s)),\ g(h(t),\ \omega t+\omega s+\theta)\rangle\, dt\,.$$

The map g being periodicaly at θ results that M is periodicaly at s of period $\frac{2\pi}{\omega}$ and periodicaly at θ of period 2π. Thus $\frac{\partial M}{\partial \theta} = \omega \frac{\partial M}{\partial s}$ and $\frac{\partial M}{\partial \theta} = 0$ if and only if $\frac{\partial M}{\partial s} = 0$.

Theorem. *Assume that there is a point (s_0,θ_0) such that:*
a) $M(s_0,\theta_0) = 0$
b) $\frac{\partial M}{\partial s}(s_0,\theta_0) \neq 0$, *then the manifolds $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ are*

intersected transversely at a point $(q_\varepsilon, \theta_0)$ *where* $d(q_\varepsilon, h(-s_0)) < \varepsilon$. *In addition, if* $M(s,\theta) \neq 0$, *for all* $(s,\theta) \in \mathbf{R} \times S^1$ *then* $W^s(\gamma_\varepsilon) \cap W^u(\gamma_\varepsilon) = \Phi$.

2. Results

Let us take the equation [1]

$$\frac{dy}{dt} = x - x^3 + \varepsilon(\gamma \sin 2\omega t - \delta y),$$

$$\text{where} \quad \frac{dx}{dt} = y \quad \gamma, \delta, \omega > 0 \quad \text{and} \quad \varepsilon > 0 \quad \text{very small.}$$

For this equation, called the Duffing oscillator modified, we have the following theorem:

Theorem. *The critical surface of the transition to chaos is* $\dfrac{\sqrt{2}\delta \mathrm{ch}\omega\pi}{3\omega\gamma\pi} = 1$, *that is, if* $\dfrac{\sqrt{2}\,\delta\,\mathrm{ch}\,\omega\pi}{3\omega\gamma\pi} < 1$ *the Duffing oscillator modified has chaotic solutions.*

Proof. For $\varepsilon = 0$, there is a saddle point at 0 in the phase plane, with two symmetric homoclinic orbits $h_1(t)$, $h_2(t)$. [2]

The orbits of the basic system (for $\varepsilon = 0$) have equation

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

It follows that the homoclinic orbits have equation

$$H(x,y) = 0, \quad \text{i.e.} \quad y^2 = x^2\left(1 - \frac{1}{2}x^2\right), \quad \text{and solutions}$$

$$h_1(t) = \left(\frac{\sqrt{2}}{\mathrm{ch}\,t}, -\frac{\sqrt{2}\cdot \mathrm{th}\,t}{\mathrm{ch}\,t}\right) \quad \text{and} \quad h_2(t) = \left(-\frac{\sqrt{2}}{\mathrm{ch}\,t}, \frac{\sqrt{2}\cdot \mathrm{th}\,t}{\mathrm{ch}\,t}\right).$$

Because $h_1(t) = -h_2(t)$, we compute the Melnikov function only for $h_1(t)$.

Thus, we have:

$$\operatorname{grad} H(x,y) = (-x + x^3,\, y) \quad \text{and}$$

$$\operatorname{grad} H(h_1(t)) = \left(\frac{-\sqrt{2}}{\mathrm{ch}\,t} + \frac{2\sqrt{2}}{\mathrm{ch}^3 t}, -\frac{\sqrt{2}\cdot \mathrm{th}\,t}{\mathrm{ch}\,t}\right)$$

$$g(x,y,\theta) = (0, n \sin 2\theta - \delta y).$$

Thus: $g(h_1(t), \omega t + \omega s + \theta) = \left(0,\, \gamma \sin 2(\omega t + \omega s + \theta) + \dfrac{\delta\sqrt{2}\cdot \mathrm{th}\,t}{\mathrm{ch}\,t}\right)$ and

$$M_1(s,\theta) = \int_{-\infty}^{\infty} \left(-\sqrt{2}\,\gamma \sin 2(\omega t + \omega s + \theta)\frac{\mathrm{th}\,t}{\mathrm{ch}\,t} - 2\delta\frac{\mathrm{th}^2\,t}{\mathrm{ch}^2\,t}\right)dt =$$

$$=\gamma\sqrt{2}\ \sin 2(\omega t+\omega s+\theta)\frac{1}{\operatorname{ch} t}\Big|_{-\infty}^{\infty}$$

$$-\,2\sqrt{2}\,\gamma\int_{-\infty}^{\infty}\omega\cos 2(\omega t+\omega s+\theta)\frac{dt}{\operatorname{ch} t}-2\delta\int_{-\infty}^{\infty}\frac{\operatorname{th}^2 t}{\operatorname{ch}^2 t}dt$$

$$=\frac{-4\delta}{3}-2\sqrt{2}\,\omega\gamma\cos 2(\omega s+\theta)\int_{-\infty}^{\infty}\frac{\cos 2\omega t}{\operatorname{ch} t}dt$$

$$+\,2\sqrt{2}\,\omega\gamma\sin 2(\omega s+\theta)\int_{-\infty}^{\infty}\frac{\sin 2\omega t}{\operatorname{ch} t}dt\,.$$

Let's note: $I_1=\int_{-\infty}^{\infty}\frac{\cos 2\omega t}{\operatorname{ch} t}dt$ and $I_2=\int_{-\infty}^{\infty}\frac{\sin 2\omega t}{\operatorname{ch} t}dt$. Then

$$I=I_1+iI_2=\int_{-\infty}^{\infty}\frac{e^{i\,2\omega t}}{\operatorname{ch} t}dt\,.$$

Using the complex theory we have:

$$I=2\pi i\Big(\sum_{k=0}^{\infty}\operatorname{rez}\Big(\frac{e^{i\,2\omega z}}{\operatorname{ch} z}\Big)\Big)\Big|_{z=z_k}=2\pi i\sum_{k=0}^{\infty}\lim_{z\to z_k}(z-z_k)\Big(\frac{e^{i\,2\omega z}}{\operatorname{ch} z}\Big)$$

where z_k are the solutions of the equation $\operatorname{ch} z=0$, for $k=0,1,2,\dots$ i.e. $z_k=\Big(\frac{\pi}{2}+k\pi\Big)i,\ k=0,1,2,\dots$

$$\text{Therefore}\quad I=2\pi\sum_{k=0}^{\infty}(-1)^k\,e^{-2\omega k\pi}\,e^{-\omega\pi}=2\pi\frac{e^{-\omega\pi}}{1-e^{-2\omega\pi}}=\frac{\pi}{\operatorname{ch}\omega\pi}$$

$$\Rightarrow\quad I_1=\frac{\pi}{\operatorname{ch}\omega\pi}\quad \text{и}\quad I_2=0\quad\Rightarrow$$

$$M_1(s,\theta)=-2\sqrt{2}\,\omega\gamma\pi\frac{\cos 2(\omega s+\theta)}{\operatorname{ch}\omega\pi}-\frac{4\delta}{3}\quad\text{and}$$

$$M_2(s,\theta)=2\sqrt{2}\,\omega\gamma\pi\frac{\cos 2(\omega s+\theta)}{\operatorname{ch}\omega\pi}-\frac{4\delta}{3}\,.$$

Using $\quad M_{1,2}=0\quad\Rightarrow\quad\cos 2(\omega s+\theta)=\pm\frac{\sqrt{2}\,\delta\operatorname{ch}\omega\pi}{3\omega\gamma\pi}\,.$

So, if we have: $\frac{\sqrt{2}\,\delta\operatorname{ch}\omega\pi}{3\omega\gamma\pi}<1$ the Duffing oscillator modified has chaotic solutions. The critical surface is $\frac{\sqrt{2}\,\delta\operatorname{ch}\omega\pi}{3\omega\gamma\pi}=1\,.$

References

[1] P. G. Drazin: *Nonlinear Systems,* Cambridge Univ. Press. 1992, p. 246-261.
[2] Nils Berglung: *Geomerycal Theory of Dynamical Systems,* Lectures Notes, 2001.
[3] P. Emilia: *Chaotic Dynamical Systems,* West Univ. Timisoara, 1992.

ПРЕМИН КОН ХАОС НА DUFFING-ОВИОТ МОДИФИЦИРАН ОСЦИЛАТОР

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Р е з и м е

Во оваа работа ние го применуваме Мелниковиот метод за модифицираниот Duffing-ов осцилатор. Ние го применивме крајниот терм на Duffing-овиот осцилатор. Ја најдовме критичната површина на преминот кон хаос.

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