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# FREE HYPERSEMIGROUPS

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Abstract: Free and strongly free objects in the class of hypersemigroups are considered, and a complete description of free objects is given. There exist a lot of free objects in the class of all hypersemigroups, but this class does not contain a strongly free object. Although three exist classes of hypersemigroups with strongly free objects, we usually have a situation where free objects do exist in a class of hypersemigroups, but a strongly free object does not exist.

**0. Preliminaries.** Here we state necessary preliminary definitions and results.

**0.1. Hypersemigroups.** A *hyperoperation* $*$ on a nonempty set $S$ is a mapping $*$: $(x, y) \mapsto x * y$ from $S^2$ into the collection of nonempty subsets of $S$. (Thus: $(\forall x, y \in S)\ (\varnothing \neq x * y \subseteq S)$. If $A, B \subseteq S$, $a, b \in S$, then $A * B$, $a * B$, $A * b$ have usual meanings, i.e.:

$$A * B = U\{x * y \mid x \in A, y \in B\},\ A * b = A * \{b\},\ a * B = \{a\} * B$$

A hyperoperation $*$ is called *associative* if:

$$(\forall x, y, z \in S)\ (x * y) * z = x * (y * z), \tag{0.1.1}$$

and then we say that $(S, *)$ is a *hypersemigroup* with a *carrier* $S$.

We will not make any distinctions between a semigroup $(S, \cdot)$ and the corresponding induced hypersemigroup $(S; *)$ defined by: $x * y = \{x \cdot y\}$.

The "general associative law" holds in any hypersemigroup $S$. In other words, if $n \geq 2$, $x_1, x_2, \ldots, x_n \in S$, then $x_1 * \ldots * x_n$ is a well defined subset of $S$.

For every nonempty set $S$, we denote by $\mathbf{H}(S)$ the set of all hypersemigroups with the same carrier $S$. The set $\mathbf{H}(S)$ is ordered in usual way, i.e. if $(S ; \circ), (S; *) \in \mathbf{H}(S)$ then:

$$(S; \circ) \leq (S; *) \Leftrightarrow (\forall x,\ y \in S)\, x \circ y \subseteq x * y, \tag{0.1.2}$$

(Then, we say that $(S; *)$ is an *expansion* of $(S; \circ)$)

The largest member $(S; \circ)$ in $\mathbf{H}(S)$ is defined by: $(\forall x,\ y \in S)\ x \circ y = S$. And $(S, \bullet)$ is a minimal member iff it is a semigroup.

**0.2. Subsemigroups.** Assume that $(S; *)$ is a hypersemigroup. A nonempty subset $R$ of $S$ is called a subsemigroup of $(S, *)$ iff $R * R \subseteq R$. A nonempty intersection of a collection of subsemigroups is a subsemigroup, as well. Hence, each nonempty subset $B$ of the carrier generates a unique subsemigroup $<B>$ of the given hypersemigroup. Namely, $<B>$ is the intersection of the collection of all subsemigroups $R$ such that $B \subseteq R$. Moreover:

$$<B> = \bigcup \{\, b_1 * \ldots * b_n \mid b_1, \ldots, b_n \in B,\quad n \geq 1 \,\} \tag{0.2.1}$$

If $S = <B>$, then $B$ *generates* $(S; *)$.

We note that if $(S, \circ) \leq (S, *)$ and $B$ generates $(S, \circ)$ then $B$ generates $(S, *)$, as well.

**0.3. Homomorphisms.** Two kinds of homomorphisms will be defined here.

Let $(S; *)$ and $(S'; *')$ be hypersemigroups and $\varphi$ a mapping from $S$ into $S'$. We say that $\varphi$: $(S: *) \to (S'; *')$ is *a homomorphisms* (a *strong homomorphism*) iff for any $x,\ y \in S$ the following relation holds:

$$\varphi(x * y) \subseteq \varphi(x) *' \varphi(y) \cdot (\varphi(x * y) = \varphi(x) *' \varphi(y)).$$

(If $A \subseteq S$, $\varphi(A)$ has the usual meaning, i.e. $\varphi(A) = \{\varphi(x) \mid x \in A\}$.)

A bijective homomorphism $\varphi$ such that $\varphi^{-1}$ is also a homomorphism, is said to be an *isomorphism*. Then $\varphi$ and $\varphi^{-1}$ are strong homomorphisms as well. Moreover, every bijective strong homomorphism is an isomorphism.

A homomorphic image of a subsemigroup is not necessarily a subsemigroup, but a nonempty complete inverse homomorphic image of a subsemigroup is a subsemigroup. A strong homomorphic image of a subsemigroup is also a subsemigroup.

If $\varphi$: $(S; *) \to (S'; *')$ is a homomorphism, and $x_1, \ldots, x_n \in S$, then

$$\varphi(x_1 * \ldots * x_n) \subseteq \varphi(x_1) *' \ldots *' \varphi(x_n),$$

and if, moreover, $\varphi$ is a strong homomorphism, then:

$$\varphi(x_1 * \ldots * x_n) = \varphi(x_1) *' \ldots *' \varphi(x_n).$$

If $(S; *)$ is an expansion of $(S; \circ)$ and $\varphi$: $(S; *) \to (S'; *')$ is a homomorphism, then $\varphi$: $(S; \circ) \to (S'; *')$ is also a homomorphism. In the case $(S; \circ) < (S; *)$ the identity mapping $1: S \to S$ is a bijective homomorphism from $(S, \circ)$ into $(S, *)$ which is not an isomorphism.

**0.4. Some Remarks.** We note that there is not any essential difference between a hyperoperation $*$ on a set $S$ and a ternary relation $\alpha$ in $S$. Namely, a biunivoque correspondence between (binary) hyperoperations and ternary relations could be established in the following way:

$$z \in x * y \Leftrightarrow (x, y, z) \in \alpha \qquad (0.4.1)$$

And, if $(S, *)$, $(S', *')$ are hypersemigroups with corresponding ternary relations: $\alpha$, $\alpha'$, then $\varphi$: $(S, *) \to (S', *')$ is a homomorphism iff

$$(x, y, z) \in \alpha \Rightarrow (\varphi(x), \varphi(y), \varphi(z)) \in \alpha' \qquad (0.4.1')$$

In other words, the notion of a homomorphism in the class of hypersemigroups is compatible with the correspondent notion of a homomorphism in the class of relation structures (See, for example, [2], p. 203, or [5], p. 11). We also note that, in most of the papers on hypersemigroups, "a homomorphism" means "a strong homomorphism" (See [1], p. 41).

**1. Free Hypersemigroups.** Let $(F; *)$ be a hypersemigroup and $B$ a nonempty subset of $F$ satisfying the following conditions:

(i) $B$ generates $(F; *)$;

(ii) If $(S, \circ)$ is a hypersemigroup and $\lambda$: $B \to S$ an arbitrary mapping, then there is a homomorphism $\varphi$: $F \to S$ which extends $\lambda$.

Then we say that $(F; *)$ is a *free hypersemigroup* with *basis B*.

Below we give a convenient description of free hypersemigroups.

**Theorem 1.1.** Let $(F; *)$ be a hypersemigroup and $B \subseteq F$. Then $(F; *)$ is a free hypersemigroup with a basis $B$ iff the following condition is satisfied:

$(\neq)$ For every $x \in F$ there is a unique sequence $b_1, b_2, \ldots, b_n$ of elements of $B$ such that:

$$x \in b_1 * \ldots * b_n \text{ [1)]} \qquad (1.1.1)$$

Moreover, $B$ is the unique basis of $(F; *)$.

1) In the case $x \in B$, (1.1.1) has the form $x = \{b_1\}$, i.e. $x = b_1$

The proof will be given in several steps.

**Lemma 1.2.** Let $B$ be a nonempty set, and

$$B^{+} = U\{B^{n} \mid n \geq 1\},$$

where $B^{1} = B$. If "$\cdot$" is the usual concatenation of strings, i.e.:

$$(b_1, \ldots, b_n) \cdot (b_{n+1}, \ldots, b_m) = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_m),$$

$(1 \leq n < m)$, then $(B^{+}, \cdot)$ is a free hypersemigroup with a basis $B$.

**Proof.** (It is well known that $(B^{+}, \cdot)$ is a free semigroup, in the class of semigroups, with a basis $B$.)

Let $(S; *)$ be a hypersemigroup and $\lambda: B \to S$ be an arbitrary mapping. If $\varphi: B^{+} \to S$ is such a mapping that

$$\varphi((b_1, \ldots, b_n)) \in \lambda(b_1) * \ldots * \lambda(b_n),$$

then $\varphi$: $(B^{+}; \cdot) \to (S; \circ)$ is a homomorphism which is an extension of $\lambda$. ¤

**Lemma 1.3.** If $B$ and $(F, *)$ satisfy $(\neq)$ then every mapping $\lambda$: $B \to B^{+}$ is the restriction of a unique strong homomorphism $\varphi$: $(F, \circ) \to (B^{+}; \cdot)$.

**Proof.** If $x \in F$, and $x \in b_1 * \ldots * b_n$, where $b_1 \ldots b_n \in B^{+}$, then $\varphi(x) = \varphi(b_1) \cdot \ldots \cdot \varphi(b_n)$. ¤

(Further on we shall usually write $b_1 \ldots b_n$ instead of $(b_1, \ldots, b_n) = v$, and say that $n = l(v)$ is the lenght of $v$.)

From **L.1.2** and **L.1.3** it follows that if $(\neq)$ holds, then $(F, *)$ is a free hypersemigroup with a basis $B$.

The uniqueness of the basis $B$ follows from the following:

**Lemma 1.4.** If $B$ and $(F, *)$ satisfy $(\neq)$, then for every $b \in B$, $F \setminus \{b\}$ is a subsemigroup of $(F; *)$.

**Proof.** Let $x, y \in F \setminus \{b\}$, where $b \in B$. Then by $(\neq)$ there exists a unique $c_1 \ldots c_n d_1 \ldots d_m \in B^{+}$ such that $m \geq 1$, $n \geq 1$ and

$$x \in c_1 * \ldots * c_n, \quad y \in d_1 * \ldots * d_m.$$

Hence:

$$b \notin c_1 * \ldots * c_n * d_1 * \ldots * d_m \supseteq x \circ y,$$

i.e.

$$x * y \subseteq \setminus \{b\}. \quad ¤$$

Assume now that $(F, *)$ is a free hypersemigroup with the basis $B$, and that $x \in F$. The fact that $B$ genarates $(F, *)$ implies that (1.1.1) holds, for some $b_1 \dots b_n \in B^+$. There is a homomorphism $\varphi$: $(F, *) \to (B^+; \cdot)$ such that $\varphi(b) = b$, for every $b \in B$. Then we have:

$$\varphi(x) \in \varphi(b_1 * \dots * b_n) = \{b_1 \dots b_n\}$$

i.e. $\varphi(x) = b_1 \cdot \dots \cdot b_n$, and this implies that there is a unique $b_1 \dots b_n \in B^+$ such that $x \in b_1 * \dots * b_n$. This completes the proof of Theorem 1.1.

**Example 1.5.** Let $B$ be a nonempty set, and $m$ a positive integer. Define a hyperoperation $\bullet$ on the set $B^+ \times N$:

$$(u, i) \bullet (v, j) = \{(uv, im^{l(v)} + \alpha m^{l(v)-1} + j) \mid 0 \leq \alpha < m\},$$

It can be easily seen that $\bullet$ is associative and, for any $b_1, \dots, b_n \in B$, $n \geq 2$, the following equality holds:

$$(b_1, 0) \bullet \dots \bullet (b_n, 0) = \{(b_1 \dots b_n, \beta) \mid 0 \leq \beta < m^{n-1}\}$$

This implies that $F = \{(u, \beta) \mid 0 \leq \beta < m^{l(u)}\}$ is a subsemigroup of $B^+ \times N$ and that $B \times \{0\}$ is a basis of $(F; \circ)$. Thus we can assume that $B$ is a basis of $(F; \circ)$. It should be also noticed that:

$$|x \circ y| = m, \quad |b_1 \bullet \dots \bullet| = m^{n-1}$$

for any $x, y \in F$, $b_1, \dots, b_n \in B$, $n \geq 2$.

**2. Special Free Hypersemigroups.** Here we consider a subclass of the class of free hypersemigroups.

**Proposition 2.1.** Let $B$ be a nonempty set and $F = B \cup C$, where $B$ and $C$ are disjoint. Let $\varphi: F \to B^+$ be a surjective mapping such that $\varphi^{-1}(b) = \{b\}$ for each $b \in B$. If a hyperoperation $*$ is defined on $F$ by:

$$x * y = \varphi^{-1}(\varphi(x)\varphi(y)) \tag{2.1}$$

for any $x, y \in F$, then $(F, *)$ is a free hypersemigroup with a basis $B$. Moreover $\varphi: F \to B^+$ is a strong homomorphism.

**Proof.** Let $x, y, z \in F$. The surjectivity of $\varphi$ implies that $x * y$ is non-emty, i.e. $*$ is a hyperoperation on $F$. If $x, y, z \in F$, then

$$x*(y*z)=\varphi^{-1}(\varphi(x)\,\varphi(\varphi^{-1}(\varphi(y)\varphi(z))) =$$
$$=\varphi^{-1}(\varphi(x)\,\varphi(y)\,\varphi(z)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(x)\,\varphi(y)))\,\varphi(z)) =$$
$$=(x*y)*z\,,$$

i. e. $*$ is associative.

Moreover, if $x_1, x_2, \ldots, x_n \in F$ then:

$$x_1 * x_2 * \ldots * x_n = \varphi^{-1}(\varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)), \tag{2.1'}$$

and this implies that:

$$b_1 * b_2 * \ldots * b_n = \varphi^{-1}(b_1 b_2 \ldots b_n), \tag{2.1''}$$

for any $b_1, \ldots, b_n \in B$. From (2.1'') it follows that $B$ generates $(F, *)$, and that the condition $(\neq)$ of Theorem 1.1 holds. ¤

The hypersemigroup $(F, *)$ defined above will be called *special* and will be denoted by $(B; \varphi)$.

**Proposition 2.2.** Two special hypersemigroups $(B; \varphi)$ and $(B'; \varphi')$ are isomorphic iff there is a bijection $\xi : B \to B'$ such that

$$|\varphi^{-1}(b_1 b_2 \ldots b_n)| = |\varphi'^{-1}(\xi(b_1)\,\xi(b_2)\ldots\xi(b_n))|\,, \tag{2.2}$$

for any $b_1, b_2, \ldots b_n \in B$, $n \geq 1$.

**Proof.** Let $\eta$ be an isomorphism from $(B; \varphi)$ onto $(B'; \varphi')$. Then $\eta(B)$ is a basis for $(B'; \varphi')$ and thus $\eta(B) = B'$, which implies that the restriction $\xi : b \to \eta(b)$ is a bijection from $B$ onto $B'$. For any $b_1, b_2, \ldots, b_n \in B$, $n \geq 2$, we have

$$\eta(b_1 * \ldots * b_n) = \eta(b_1) * \eta(b_2) * \ldots * \eta(b_n) = \xi(b_1 * \ldots * \xi(b_n),$$

and thus the restriction of $\eta$ on $b_1 * \ldots * b_n$ induces a bijection:

$$\xi : \varphi^{-1}(b_1 \ldots b_n) = b_1 * \ldots * b_n \to \xi(b_1) * \ldots * \xi(b_n) = \varphi'^{-1}(\xi(b_1)\ldots\xi(b_n))$$

Therefore (2.2) holds for any $b_1, \ldots, b_n \in B$.

Conversely, assume that $\xi : B \to B'$ is a bijection such that (2.2) holds for any $b_1, \ldots, b_n \in B$, $n \geq 1$. This implies that there is a bijection

$$\eta_{b_1 \ldots b_n} : \varphi^{-1}(b_1 \ldots b_n) \to \varphi'^{-1}(\xi(b_1)\ldots\xi(b_n)).$$

Then:

$$\eta = \cup \left\{ \eta_{b_1 \ldots b_n} \mid b_1, \ldots, b_n \in B, \ n \geq 1 \right\}$$

is an isomorphism from $(B, \varphi)$ onto $(B', \varphi)$, such that $\eta(b) = \xi(b)$, for any $b \in B$. ¤

We will show now that every free hypersemigroup admits an expansion which is a special hypersemigroup.

**Proposition 2.3.** Let $(F\,; \mathrm{o})$ be a free hypersemigroup with a basis $B$, and let a hyperoperation $*$ be defined on $F$ as follows:

$$x \in b_1 \mathrm{o} \ldots \mathrm{o}\, b_n, \ y \in c_1 \mathrm{o} \ldots \mathrm{o}\, c_m, \ (b_i, c_j \in B, n, m \geq 1) \Rightarrow$$

$$x * y = b_1 \mathrm{o} \ldots \mathrm{o}\, b_n \mathrm{o}\, c_1 \mathrm{o} \ldots \mathrm{o}\, c_m \tag{2.3}$$

Then, $(F, *)$ is a special hypersemigroup such that $(F\,; \mathrm{o}) \leq (F, *)$, and moreover the equality

$$b_1 * \ldots * b_n = b_1 \mathrm{o} \ldots \mathrm{o}\, b_n, \tag{2.3'}$$

holds, for any $b_1, \ldots, b_n \in B$, $n \geq 1$.

**Proof.** By Th. 1.1, if $x \in F$, then there is a unique $b_1 \ldots b_n \in B^+$ such that $x \in b_1 \mathrm{o} \ldots \mathrm{o}\, b_n$. (In the case $n = 1$, we hawe $x = b_1 \in B$.) We can define a surjective mapping $\varphi : F \to B^+$ in the following way:

$$\varphi(x) = b_1 \cdot \ldots \cdot b_n \Leftrightarrow x \in b_1 \mathrm{o} \ldots \mathrm{o}\, b_n, \tag{2.3''}$$

and moreover we have $\varphi^{-1}(b) = \{b\}$, for any $b \in B$. Thus we have a special hypersemigroup $(F\,; *) = (B; \varphi)$. By (2.1) and (2.3'') it can be easily seen that (2.3) holds, and (2.3') is a "translation" of (2.1'').

If $x, y$ are as in (2.3) then:

$$x \mathrm{o}\, y \subseteq b_1 \mathrm{o} \ldots \mathrm{o}\, b_n \mathrm{o}\, c_1 \mathrm{o} \ldots \mathrm{o}\, c_m = x * y,$$

and therefore $(F, \mathrm{o}) \leq (F, *)$. ¤

Now we can give an intrinsic description of special hypersemigroups.

**Proposition 2.4.** A free hypersemigroup $(F, \mathrm{o})$ with a basis $B$ is a special hypersemigroup iff the following condition is satisfied.

($\neq\neq$) If $b_1, \ldots, b_n, \ c_1, \ldots, c_m \in B, \ n, m \geq 1$ then:

$$x \in b\, \mathrm{o} \ldots \mathrm{o}\, b_n, \ y \in c\, \mathrm{o} \ldots \mathrm{o}\, c_m \Rightarrow x * y = b_1 \mathrm{o} \ldots \mathrm{o}\, b_n \mathrm{o}\, c_1 \mathrm{o} \ldots \mathrm{o}\, c_m$$

**Proof.** From Pr. 2.1 it folows that $(\neq\neq)$ is satisfied in any special hypersemigroup.

And conversely, if $(F; \circ)$satisfy $(\neq\neq)$then $(F, \circ) = (F, *)$, where $(F, *)$ is the special hypersemigroup defined in Pr. 2.3. ¤

As a corollary we obtain the following two statements:

**Proposition 2.5.** The class of special hypersemigroups is abstract. (In other words, if $(F, *)$ and $(F'; *')$ are two isomorphic hypersemigroups one of which is a special, then both of them are special.) ¤

**Proposition 2.6.** If $B$ is a nonempty set and $\tau$ a cardinal such that $\tau \geq \max\{|B|, \aleph_o\}$, then there exists a special hypersemigroup $(F, *)$ such that $|F| = \tau$. ¤

( As usual, we denote by $|S|$ the cardinality of a set $S$.)

The existence of free hypersemigroups that are not special follows from Example 1.6. (Namely, in the case $m \geq 2$, $(F, \bullet)$ is a non special free hypersemigroup.)

It should be noted that if $(F, \circ) < (B; \varphi)$ where $B$ generates $(F, \circ)$ then $B$ is a basis of $(F, \circ)$ and $(F, \circ)$ is not special.

Below, we describe the class of special hypersemigroups which are expansions of nonspecial free hypersemigroups.

**Proposition 2. 7.** A special hypersemigroup $(B; \varphi)$ is an expansion of a nonspecial free hypersemigroup $(F, \circ)$ iff the following condition is satisfied:

$(\neq\neq\neq)$ There exist $u, v \in B^+$ such that

$$(|\varphi^{-1}(u)| \geq 2 \text{ or } |\varphi^{-1}(v)| \geq 2) \text{ and } |\varphi^{-1}(uv)| \geq 2.$$

**Proof.** Assume that $u, v \in B^+$ are such that:

$$\varphi^{-1}(u) = C' \cup C'', \ \varphi^{-1}(uv) = D' \cup D'',$$

where:

$$C', C'', D', D'' \neq 0, \quad C' \cap C'' = 0, \quad D' \cap D'' = 0$$

Define a hyperoperation $\circ$ on $F$ as follows. If $x' \in C'$, $x'' \in C''$, $y \in \varphi^{-1}(v)$ then $x' \circ y = D'$, $x'' \circ y = D''$, and $x \circ y = \varphi^{-1}(\varphi(x)\varphi(y))$ $(= x * y)$, in all other cases. Then $(F, \circ)$ is a nonspecial free hypersemigroup with a basis $B$, and $(B; \varphi)$ is an expansion of $(F, \circ)$.

Conversely, assume that $(B;\varphi)$ is an expansion of a nonspecial free semigroup $(F,\circ)$ with a basis $B$. Then, there exist $x^*, y^* \in F$ such that $x^*\circ y^* \subset \varphi^{-1}\left(\varphi(x^*)\,\varphi(y^*)\right)$. Thus if

$$\varphi(x^*) = u = a_1 a_2 \dots a_n,\ \varphi(y^*) = b_1 b_2 \dots b_m,\ a_i, b_j \in B,$$

then:

$$x^* \in a_1\circ \dots \circ a_n,\ y^* \in b_1\circ b_2\circ \dots \circ b_m,\ x^*\circ y^* \subset a_1\circ \dots \circ a_n \circ b_1 \dots \circ b_m$$

Thus $|\varphi^{-1}(uv)| \geq 2$, where $\varphi(x) = u$, $\varphi(y) = v$. Moreover, we have:

$$\varphi^{-1}\left(\varphi(x)\,\varphi(y)\right) = \cup\left\{x\circ y \mid x \in \varphi^{-1}(u),\ y \in \varphi^{-1}(v)\right\},$$

and this implies that $|\varphi^{-1}(u)| \geq 2$ or $|\varphi^{-1}(v)| \geq 2$. Hence $(\neq\neq\neq)$ holds.¤

As an illustration consider the case $|B| = 1$, i.e. $B = \{b\}$, $B^+ = \{b^n \mid n \geq 1\}$. Then the condition $(\neq\neq\neq)$ does not hold iff there is a $k \geq 2$ such $|\varphi^{-1}(b^k)| \geq 2$ and $|\varphi^{-1}(b^m)| = 1$, for any $m \neq k$. In this case we can assume that $F = (B^+ \setminus \{b^k\}) \cup C$, where $C \cap (B^+ \setminus \{b^k\}) = 0$ and $|C| \geq 2$.

**3. Strongly free hypersemigroups.** The fact that we have two kinds of homomorphisms implies that we can define two kinds of free hypersemigroups. But free hypersemigroups corresponding to strong homomorphisms would not exist as can be seen from the following statement.

**Proposition 3.1.** Let $S$ and $T$ be two nonempty sets, such that $|S| < |T|$. If $(T;\bullet)$ is the largerst member of $\mathbf{H}(T)$, i.e. $(\forall x, y \in T)\, x \bullet y = T$ and $(S;\circ) \in \mathbf{H}(S)$, then there is not a strong homomorphism from $(S;\circ)$ into $(T;\bullet)$. ¤

This result suggests considering a strongly free object in a class $\mathbf{C}$ of hypersemigroups. Namely, a hypersemigroup $(F;*) \in \mathbf{C}$ is called a *strongly free object* in $\mathbf{C}$ iff there is a nonempty subset $B$ of $F$ such that the following conditions hold:

(i) $B$ generates $(F;*)$;

(ii) If $(S;\circ) \in \mathbf{C}$ and $\lambda$ is a mapping from $B$ into $S$, then there is a strong homomorphism $\varphi: (F;*) \to (S;\circ)$ which is an extension of $\lambda$.

Then we also say that $B$ is a *strong* $\mathbf{C}$*-basis* of $(F;*)$.

Certainly, the meaning of the following statement is clear: "$(F;*)$ is a free object in $\mathbf{C}$ and $B$ is a $\mathbf{C}$-basis in $(F;*)$". Moreover, every strongly free object in a class **C** is a free object in **C**, as well. Thus in a class of semigroups "strongly free" has the same meaning as "free".

A corollary of **Pr. 3.1.** is the following statement:

**Proposition 3.2.** There does not exist a strongly free object in the class **H** of all the hypersemigroups. ¤

Below we assume that **C** is a class of hypersemigroups that satisfies the following conditions:

(iii) Every free semigroup is in **C**.

(iv) If $(S; \circ) \in \mathbf{C}$ then $x \circ y$ is finite for any $x, y \in S$.

It can be easily seen that all previously obtained results on free objects are also true for free objects in such a class **C**.

The next three statements concern an arbitrary class **C** of hypersemigroups that satisfies (iii) and ( iv).

**Proposition 3.3.** If $(S; \circ) \in \mathbf{C}$ and $x_1, \ldots, x_n \in S$, $n \geq 2$, then $x_1 \circ \ldots \circ x_n$ is finite. ¤

**Proposition 3.4.** If $(F; \circ)$ is a free object in **C** with a **C**-basis $B$ and $\xi$ is a strong endomorphism of $(S; \circ)$, then $\xi$ is an automorphism.

**Proof.** If $b_1, b_2, \ldots, b_n \in B$, $n \geq 2$, then:

$$\xi(b_1 \circ \ldots \circ b_n) = \xi(b_1) \circ \ldots \circ \xi(b_n) = b_1 \circ \ldots \circ b_n,$$

and thus $\xi$ induces a permutation of $b_1 \circ \ldots \circ b_n$. Using the condition $(\neq)$ we obtain that $\xi$ is bijective, i.e. an automorphism. (We recall that every strong bijective homomorphism is an isomorphism.) ¤

**Theorem 3.5.** If $(F; *)$, $(F'; *')$ are strongly free objects in **C** with a same **C**-basis $B$, then they are isomorphic.

**Proof.** Let $\xi: (F; *) \to (F'; *')$, $\eta: (F'; *') \to (F; *)$ be strong homomorphism such that $(\forall\, b \in B)\ \xi(b) = \eta(b) = b$. From **Pr. 3.4** it follows that $\zeta = \eta\xi$ is an automorphism of $(F; *)$, and therefore $\xi$ is a bijective strong homomorphism, i.e. an isomorphism. ¤

Below we consider the following classes of hypersemigroups $\mathbf{H}[m]$, $\mathbf{H}[[m]]$, where $m$ is a positive integer.

$$(S; \circ) \in \mathbf{H}[m] \Leftrightarrow (\forall x_1, \ldots, x_n \in S)\ \ |x_1 \circ \ldots \circ x_n| \leq m$$

$$(S; \circ) \in \mathbf{H}[[m]] \Leftrightarrow (\forall x, y \in S)\ \ |x \circ y| \leq m.$$

Clearly $\mathbf{H}[m]$ and $\mathbf{H}[[m]]$ satisfies (iii) and ( iv). (Note that $\mathbf{H}[1] = \mathbf{H}[[1]]$, is the class of semigroups.)

**Theorem 3.6.** If $m \geq 2$, then the class of strongly free objects in $\mathbf{H}[m]$ is empty.

**Proof.** Assume that $(F; *)$ is strongly free in $\mathbf{H}[m]$ with a strong $\mathbf{H}[m]$-basis $B$. If $(B; \varphi)$ is a special hypersemigroup such that $|\varphi^{-1}(u)| = m$, for every $u \in B^+ \setminus B$, then $(B; \varphi) \in \mathbf{H}[m]$, and thus there is a strong homomorphism $\xi: (F; *) \to (B; \varphi)$ such that $\xi(b) = b$ for each $b \in B$. In the same way as in the proof of **Pr. 3.4** it can be obtained that $\xi$ is bijective, i.e. an isomorphism. Hence $(F; *)$ is a special hypersemigroup $(B; \psi)$ such that $|\psi^{-1}(u)| = m$, for every $u \in B^+ \setminus B$.

Let $a$ be a fixed element of $B$, and let:

$$\psi^{-1}(a^2) = C' \cup C'', \quad \psi^{-1}(a^3) = D' \cup D'',$$

where $C', C'', D', D''$ are nonempty and $C' \cap C'' = D' \cap D'' = 0$.

Define a hyperoperation $\circ$ on $F$ as follows:

$$(\forall b \in B,\ x' \in C',\ x'' \in C'')\ b \circ x' = x' \circ b = D'$$
$$b \circ x'' = x'' \circ b = D''$$

and $x \circ y = x * y$ in all other cases.

Then $(F; \circ) \in \mathbf{H}[m]$. (In fact $(F; \circ)$ is a nonspecial free object in $\mathbf{H}[m]$ an expansion of which is $(F; *)$.)

Let $\xi: (F; *) \to (F; \circ)$ be a strong homomorphism such that $(\forall b \in B)$ $\xi(b) = b$. Then:

$$\xi(a * a) = \xi(a) \circ \xi(a) = a \circ a = C' \cup C''$$

Hence, if $x' \in C'$, $x'' \in C''$ there exist $y, z \in a * a$ such that $\xi(y) = x'$, $\xi(z) = x''$. This implies:

$$D' = a \circ x' = \xi(a) \circ \xi(y) = \xi(a * y) = \xi(a * a * a) = \xi(a * z) =$$
$$= \xi(a * a * a) = \xi(a * z) = \xi(a) \circ \xi(z) = a \circ x'' = D'',$$

and this is a contradiction. ¤

By analyzing the last proof we obtain that the following statement is also true.

**Proposition 3.7.** If $m \geq 2$ then the class of free objects in $\mathbf{H}[m]$ does not contatin a strongly free object. ¤

Below, we consider two subclasses of $\mathbf{H}[m]$ with strongly free objects.

**Proposition 3. 8.** If $\mathbf{S}[m]$ is the class of special objects in $\mathbf{H}[m]$ then $(B;\varphi)$ is a strongly free object in $\mathbf{S}[m]$ iff

$$(\forall u \in B^{+}\backslash B) \ |\varphi^{-1}(u)| = m. \qquad (*)$$

**Proof.** Assume that $(B;\varphi) = (F;*)$ satisfies (*), and that $\lambda: B \to F'$ is an arbitrary mapping, where $(F';*') \in \mathbf{S}[m]$. If $u = b_1 b_2 \dots b_n$, $n \geq 2$, $b_i \in B$ then there is a surjective mapping

$$\xi_u : b_1 * \dots * b_n = \varphi^{-1}(u) \to \lambda(b_1) *' \dots *' \lambda(b_n),$$

and $\xi = \cup\{\xi_u \mid u \in B^{+}\}$ is a strong homomorphism from $(F;*)$ into $(F';*')$.¤

Below we consider an other subclass $\mathbf{L}[m]$ of $\mathbf{H}[m]$. Namely, $\mathbf{L}[m]$ consists of all the objects $(S;\circ) \in \mathbf{H}[m]$ which satisfy the following identity equations:

$$x \circ y = y \circ x, \quad x \circ y \circ z = (x \circ y) \cup (x \circ z) \qquad (3.1)$$

The following statement can be easily shown:

**Proposition 3.9.** If $(S;\circ) \in \mathbf{L}[m]$ then the following implication holds:

$$\{x_1, \dots, x_p\} = \{y_1, \dots, y_q\} \Rightarrow x_1 \circ \dots \circ x_p = y_1 \circ \dots \circ y_q, \qquad (3.2)$$

for any $x_1, \dots, x_p$, $y_1, \dots, y_q \in S$, $p, q \geq 2$. ¤

For any nonempty set $S$ we denote by $F(S)$ the collection of nonempty finite subsets of $S$.

**Proposition 3.10.** If $(S;\circ) \in \mathbf{L}[m]$, and if for any $X = \{x_1, \dots, x_n\} \subseteq S$, $n \geq 2$, $f(X)$ is defined by:

$$f(X) = x_1 \circ \dots \circ x_n, \qquad (3.3)$$

then $f$ is a transformation of $F(X)$, such that:

$$|f(X)| \leq m, \ f\big(f(X) \cup Y\big) = f(X \cup Y), \qquad (3.4)$$

for any $X$, $Y \in F(S)$. ¤

A transformation $f$ of $F(S)$ which satisfies (3.4), in ([3], p. 77) is said to be an *associative* m-*object* $(S;f)$.

**Proposition 3.11.** Let $(S;f)$ be an associative $m$-object and let a hyperoperation $\circ$ be definef on $S$ by:

$$x \circ y = f(\{x, y\}), \qquad (3.5)$$

for any $x$, $y \in S$. Then $(S\,;\circ) \in \mathbf{L}\,[m]$, and moreover (3.3) is satisfied for any $X = \{x_1, \ldots, x_n\} \in F(S)$, $n \geq 2$. ¤

(We say that $(S; f)$ and $(S; \circ)$ are associated.)

The notions of subobjects and homomorphisms, in the class of associative $m$-objects are defined in [3] as usual, and they induce the corresponding notions of free associative $m$-objects. Moreover, the following statement holds:

**Proposition 3.12.** If $(S; f)$ is a free associative m-object with a basis $B$ the corresponding associated hypersemigroup $(S; \circ)$ is strongly free in $\mathbf{L}\,[m]$ with a strong $\mathbf{L}\,[m]$-basis $B$. ¤

The following statement is a translation of **Pr. 2.9 – 2.11** in the paper mentioned ([3], p. 82–84).

**Theorem 3.13.** Every nonempty set is a strong $\mathbf{L}\,[m]$-basis of a strongly free object in $\mathbf{L}\,[m]$, and two strongly free objects in $\mathbf{L}\,[m]$ with the same $\mathbf{L}\,[m]$-basis are isomprphic. ¤

We finish our discussion stating the following result:

**Proposition 3.14.** If there exists a strongly free object in $\mathbf{H}\,[[m]]$ with a strong $\mathbf{H}\,[[m]]$-basis $B$, then it is isomorphic with the hypersemigroup $(F; \bullet)$ of Example 1.5. ¤

We note that the question of the existence of a strongly free object in $\mathbf{H}\,[[m]]$ is still open.

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Резиме

**СЛОБОДНИ ХИПЕРПОЛУГРУПИ**

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Се изучуваат слободните и јако слободните објекти во класата хиперполугрупи, при што се дава комплетен опис на слободните објекти. Класата од сите хиперполугрупи е богата со слободни објекти, но во оваа класа не постои јако слободен објект. И покрај тоа што постојат класи хиперполугрупи со јако слободни објекти, обично во една класа хиперполугрупи има слободни објекти, а ретки се класите што имаат јако слободни објекти.

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