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WEAKLY COMMUTATIVE n-SEMIGROUPS

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Abstract. In this note we generalize the notion of weakly commutative semigroup to the n-ary case and we show that every weakly commutative n-semigroup is an n-semilattice of archimedean n-semigroups.

1. Some definition and results. If $(x_1,\ldots,x_n) \to x_1x_2\ldots x_n$ is an associative n-ary operation on a set S, then we say that S is an *n-semigroup*. A subset A of S is called an *i-ideal* of S iff $S^{i-1}AS^{n-i} \subsetneq A$, if i=n, then A is called a *left ideal*, and if i=1 a *right ideal*; A is *two-sided ideal* iff it is a a left and right ideal; A is an *ideal* iff it is an i-ideal for every $i\in\{1,2,\ldots,n\}$. An n-semigroup S is said to be *two sided simple* (*left-simple*) if it has no proper two sided ideal (left ideal). An ideal A is said to be *completely prime* iff: $x_1x_2\ldots x_n \in A \Leftrightarrow$ $\Leftrightarrow (\forall i) x_i \in A$. A *filter* of S is any subset $B \subsetneq S$ which complement in S is a completely prime ideal. If x is a given element of S, then N(x) will denote the intersection of all fillters of S which contain x.

An n-semigroup S is called an *n-band* iff it is idempotent, i.e. iff the identity $x^n=x$ holds in S. If, in adition, S is commutative and for every $i_\nu, j_\nu > 0$ such that

$$i_1+i_2+\ldots+i_k = j_1+j_2+\ldots+j_k = n,$$

the identity

$$x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k} = x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k}$$

holds in S, then we say that S is an *n-semilattice*. A congruence α in S is called a *semilattice congruence* iff S/α is an n-semilattice.

We will formulate some results proved in [2] and [3]. Throughout the paper, S will denote a given n-semigroup if it is not said otherwise.

1.1. ([2] 3.2) The relation η defined in S by

$$x\eta y \iff N(x) = N(y)$$

is the minimal semilattice congruence on S.

(If $x\in S$, then the η-class which contains x will be denoted by N_x.)

A constructive way for obtaining N(x), which has an inductive nature, is given in ([2] 3.4).

1.2. ([3] 2.2) If $x\in S^{n-1}xS^{n-1}$ for every $x\in S$, then

$$N_x = \{y\in S \mid x\in S^{n-1}yS^{n-1},\ y\in S^{n-1}xS^{n-1}\}$$

also for every $x\in S$.

2. We will now consider weakly commutative n-semigroups where N(x) has a simple forme.

An n-semigroup S is weakly commutative iff for any $x_1,x_2,\dots,x_n\in S$,

$$(x_1x_2\dots x_n)^{k(n-1)+1}\in x_2x_3\dots x_nS^{n-1}x_1\cap x_3x_4\dots x_nS^{n-1}x_1x_2\cap\dots$$
$$\dots\cap x_nS^{n-1}x_1x_2\dots x_{n-1}.$$

2.1. If S is a weakly commutative n-semigroup, then for every $x\in S$

$$N(x) = \{y\in S \mid x^{k_1(n-1)+1}\in S^{n-1}y \text{ for some } k_1\in N\} =$$
$$= \{y\in S \mid x^{k_2(n-1)+1}\in yS^{n-1} \text{ for some } k_2\in N\}.$$

Proof. For $x\in S$, let

$$T = \{y\in S \mid x^{k(n-1)+1}\in S^{n-1}y \text{ for some } k\in N\}.$$

We show first that T is a filter of S. Let $y_1,y_2,\dots,y_n\in T$. Then

$$x^{k_1(n-1)+1} = a_{11}a_{12}\ldots a_{1n-1}y_1$$
$$x^{k_2(n-1)+1} = a_{21}a_{22}\ldots a_{2n-1}a_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x^{k_n(n-1)+1} = a_{n1}a_{n2}\ldots a_{nn-1}y_n$$

for some $a_{11},a_{12},\ldots,a_{nn-1}\in S$ and some $k_1,k_2,\ldots,k_n\in N$. Since S is weakly commutative n-semigroup, we have

$$(a_{21}a_{22}\ldots a_{2n-1}y_2)^{r_2(n-1)+1} = y_2b_{21}b_{22}\ldots b_{2n-1}$$

for some $b_{21},b_{22},\ldots,b_{2n-1}\in S$ and some $r_2\in N$. Therefore

$$x^{k_1(n-1)+1}(x^{k_2(n-1)+1})^{r_2(n-1)+1}\ldots x^{k_m(n-1)+1} =$$
$$= a_{11}a_{12}\ldots a_{1n-1}y_1(y_2b_{21}b_{22}\ldots b_{2n-1})\ldots a_{n1}a_{n2}\ldots a_{nn-1}y_n.$$

Hence again by weak commutativity of S, there exist $r_3\in N$, $b_{31},b_{32},\ldots,b_{3n-1}\in S$ such that

$$(b_{21}b_{22}\ldots b_{2n-1}a_{31}a_{32}\ldots a_{3n-1}y_3)^{r_3(n-1)+1} =$$
$$= y_3b_{31}b_{32}\ldots b_{3n-1}$$

Continuing this process, we obtain afther n-3 steps

$$x^{t(n-1)+1} = c_1c_2\ldots c_{n-1}y_1y_2\ldots y_n\in S^{n-1}y_1y_2\ldots y_n$$

and thus $y_1y_2\ldots y_n\in T$.

Conversely, suppose that $y_1y_2\ldots y_n\in T$; then there exist some $m\in N$ and some $a_{11},a_{12},\ldots,a_{1n-1}\in S$ such that

$$x^{m(n-1)+1} = a_{11}a_{12}\ldots a_{1n-1}y_1y_2\ldots y_n\in S^{n-1}y_n$$

and thus $y_n\in T$.

Since S is weakly commutative, there exist some $k_1\in N$ and some $b_{11},b_{12},\ldots,b_{1n-1}$ such that

$$[(a_{11}a_{12}\ldots a_{1n-1}y_1)y_2\ldots y_n]^{k_1(n-1)+1} = b_{11}b_{12}\ldots b_{1n-1}y_1\in S^{n-1}$$

and hence $y_1\in T$. Similarly we can prove that $y_2,\ldots,y_{n-1}\in T$. Therefore T is a filter.

Since $x^n \in S^{n-1}x$, we have $x \in T$, by minimality of $N(x)$, it must be $N(x) \subsetneq T$. On the other hand $T \subsetneq N_2(x) \subsetneq N(x)$, so that $T=N(x)$.

An n-semigroup S is archimedean if for any $a,b \in S$ there exists a positive integer k for which $a^{k(n-1)+1} \in S^{n-1}bS^{n-1}$.

2.2. Every weakly commutative n-semigroup is an n-semilattice of archimedean n-semigroups.

Proof. We show that N_x is archimedean. Let $a,b \in N_x$ then $N(a)=N(b)=N(x)$, $ba^{n-1} \in N(x)$.

By this is follows

$$N(ba^{n-1}) \subsetneq N(x).$$

On the other hand,

$$a \in N(ba^{n-1}), \text{ it implies } N(a) \subsetneq N(ba^{n-1}), \text{ i.e.}$$

$$N(x) \subsetneq N(ba^{n-1}). \text{ Therefore } N(ba^{n-1}) = N(x).$$

By 2.1, $N(x)=\{y \in S \mid x^{k(n-1)+1} \in yS^{n-1}\}$ and we obtain that $ba^{n-1} \in N(a)$, which means that $ba^{n-1}t_1t_2\ldots t_{n-1} \in N(a)$, i.e. $at_1t_2\ldots t_n \in N(a)$. This implies that

$$N(at_1t_2\ldots t_n) \subsetneq N(a).$$

Since $a \in N(at_1t_2\ldots t_n)$ it follows that

$$N(a) = N(a.t_1t_2\ldots t_{n-1}) = N(x), \text{ i.e. } at_1t_2\ldots t_{n-1} \in N_x$$

Since $ba^{n-1}t_1t_2\ldots t_{n-1} \in N(a)$, we have that

$$a^{k(n-1)+1} = ba^{n-1}t_1t_2\ldots t_{n-1}$$

$$a^{n-1}a^{k(n-1)+1} = a^{n-1}ba^{n-1}.t_1t_2\ldots t_{n-1}, \text{ i.e.}$$

$$a^{(k+1)(n-1)+1} \in N_x^{n-1}bN_x^{n-1}$$

We obtain that N_x is an archemedean n-semigroup, i.e. S is an n-semilattice of archemedean n-semigroups.

R E F E R E N C E S

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СЛАБО КОМУТАТИВНИ n-ПОЛУГРУПИ

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Р е з и м е

Во о ваа работа се обопштуваат слабо комутативните полугрупи за n-арен случај. Се покажува дека секоја слабо комутативна n-полугрупа е n-полумрежа од архимедови n-полугрупи.